On limit theorems and backward stochastic differential equations via Malliavin calculus

Solesne Bourguin

Université Paris 1 Panthéon Sorbonne

Ecole Doctorale Paris Centre

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Solesne Bourguin

Sur les théorèmes limites et les équations différentielles stochastiques rétrogrades par le calcul de Malliavin

dirigée par Ciprian Tudor

Soutenue le 13 décembre 2011 devant le jury composé de :

M. Vlad Bally	Université de Marne-la-Vallée	rapporteur
M. Emmanuel Gobet	Ecole Polytechnique	examinateur
M^{me} Annie Millet	Université de Paris 1	examinateur
M. Ivan Nourdin	Université de Nancy 1	examinateur
M. David Nualart	The University of Kansas	rapporteur
M. Giovanni Peccati	Université du Luxembourg	examinateur
M. Ciprian Tudor	Université de Lille 1	directeur
M. Frederi Viens	Purdue University	examinateur

Laboratoire SAMM, EA 4543, Université Paris 1 Panthéon Sorbonne, 90 Rue de Tolbiac, 75634 Paris Cedex France

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Résumé

Résumé

Cette thèse, composée de trois parties, est centrée sur l'application du calcul de Malliavin à différents domaines de l'analyse stochastique, tels que les théorèmes limites, le calcul stochastique fractionnaire et la régularité des solutions d'équations différentielles stochastiques. La première partie porte sur l'étude asymptotique de modèles de regression fractionnaire et fait appel au calcul stochastique par rapport au mouvement Brownien fractionnaire et au calcul de Malliavin. La deuxième partie est centrée sur la méthode de Stein sur l'espace de Wiener et présente des résultats ayant attrait aux théorèmes limites pour des fonctionnelles de champs Gaussiens (processus moyenne mobile à mémoire longue, sommes autonormalisées) ainsi que des résultats portant sur des propriétés de déconvolution de la loi Gamma. La troisième et dernière partie a pour objet l'étude, par le calcul de Malliavin, des solutions d'équations différentielles stochastiques rétrogrades, et en particulier l'existence de densité ainsi que d'estimées de densité pour ces solutions.

Mots-clefs

Calcul de Malliavin, théorèmes limites, mouvement Brownien fractionnaire, modèles de régression, moyennes mobiles à mémoire longue, théorèmes de Cramér, loi Gamma, sommes autonormalisées, équations différentielles stochastiques rétrogrades, estimées de densités.

On limit theorems and backward stochastic differential equations via Malliavin calculus

Abstract

This thesis is organized in three distinct parts, all of which focus on the application of the Malliavin calculus to various areas of stochastic analysis such as limit theorems, fractional stochastic calculus and regularity of the solutions to stochastic differential equations. The first part is dedicated to the asymptotic study of fractional regression models via Malliavin calculus and stochastic calculus with respect to fractional Brownian motion. The second part deals with Stein's method on the Wiener space and several results on limit theorems for functionals of Gaussian fields (long memory moving averages, self-normalized sums) are presented, along with results on the deconvolution properties of the Gamma distribution. The third and last part addresses the study of the solutions to stochastic differential equations and backward stochastic differential equations and more precisely the study of the conditions for those to have a density for which upper and lower estimates can be derived.

Keywords

Malliavin calculus, limit theorems, fractional Brownian motion, regression models, long memory moving averages, Cramér theorems, Gamma distribution, self-normalized sums, backward stochastic differential equations, density estimates.

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Chapitre 0

Introduction

Cette thèse est divisée en trois parties distinctes mais ayant toutes comme dénominateur commun le calcul de Malliavin. Le calcul de Malliavin, connu aussi sous le nom de *calcul des variations stochastique*, est un calcul différentiel en dimension infinie introduit par Paul Malliavin en 1976 [Mal78] pour étudier la régularité des solutions d'équations différentielles stochastiques. La notion de différentiation pour des processus stochastiques a contribué à compléter la théorie de l'intégration d'Itô. Ce calcul des variations stochastique a également permis de donner plus tard une preuve probabiliste du théorème d'Hörmander sur les opérateurs différentiels hypoelliptiques. Le cadre d'application du calcul de Malliavin s'est ensuite considérablement étendu dans les décennies qui ont suivies. Il a, par exemple, permit d'introduire le calcul stochastique anticipant et plus particulièrement le calcul stochastique par rapport à des processus fractionnaires. Les premières applications en mathématiques financières du calcul de Malliavin remontent à la fin des années 1990 avec l'article de Fournié et al. [FLL+99] qui a sucisté un grand engouement pour ces techniques dans le monde des mathématiques appliquées. Récemment, suite aux travaux fondateurs de Nourdin et Peccati [NP09c] et de Nualart et Peccati [NP05], le calcul de Malliavin est entré au coeur de l'étude des théorèmes limites, et plus particulièrement de l'étude de la vitesse de convergence dans ces derniers. Cette thèse concerne quasiment tous les domaines d'application du calcul de Malliavin. En effet, il y est question de calcul stochastique fractionnaire, de théorèmes limites mais également de l'étude de la régularité d'équations différentielles stochastiques.

Cette thèse est composée des cinq articles suivants :

S. Bourguin et C.A. Tudor : Asymptotic theory for fractional regression models via Malliavin calculus. *J. Theoret. Probab.*, 2010. A paraître.

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O. Aboura et S. Bourguin : Density estimates for solutions to one dimensional sde's and backward sde's. *Soumis pour publication à Potential Analysis*, 2011.

- ▷ La première partie de cette thèse est consacrée à l'étude de la convergence de suites de variables aléatoires qui ne sont pas des semi-martingales apparaissant naturellement dans l'étude asymptotique de modèles de régression fractionnaires.
- ▷ La deuxième partie de cette thèse est dédiée à des applications de la méthode de Stein sur l'espace de Wiener introduite par Nourdin et Peccati dans [NP09c]. Le but initial de la méthode de Stein est de mesurer l'éloignement entre des lois de probabilité au moyen d'opérateurs caractéristiques des lois considérées et de l'étude des solutions de l'équation de Stein associée. Plus tard, ces techniques probabilistes ont été combinées au calcul de Malliavin par Nourdin et Peccati [NP09c], permettant notamment l'obtention de bornes des vitesses de convergence dans les théorèmes limites sur l'espace de Wiener. Ces bornes font intervenir les opérateurs différentiels du calcul de Malliavin. Dans cette thèse, les applications de la méthode de Stein et du calcul de Malliavin portent notamment sur l'obtention de bornes de Berry-Esséen dans des théorèmes de la limite centrale ainsi que sur des propriétés nouvelles de la loi Gamma.
- ▷ Dans la dernière partie de cette thèse, nous utilisons le calcul de Malliavin dans le cadre de sa vocation initiale, à savoir l'étude de la régularité des lois de probabilité des solutions d'équations différentielles stochastiques et d'équations différentielles stochastiques rétrogrades. Plus précisement, nous nous intéressons aux conditions nécessaires à l'existence de densité pour ces solutions, ainsi qu'aux conditions requises pour encadrer ces densités par des estimées, gaussiennes ou plus générales. Tous ces travaux étant tous fortement basés sur le calcul de Malliavin, nous commencerons par une introduction plus détaillée de ce calcul différentiel sur l'espace de Wiener.

0.1 Calcul de Malliavin

Considérons un processus gaussien centré $(B_t)_{t\in[0,T]}$ définit sur un espace probabilisé $(\Omega, \mathcal{F}, \mathbf{P})$ où $\mathcal{F}$ est la tribu engendrée par B. Soit $\mathcal{E}$ l'espace des fonctions simples sur l'intervalle $[0,T]$. On définit alors l'espace de Hilbert $\mathfrak{H}$ comme la fermeture de $\mathcal{E}$ par rapport au produit scalaire

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = \mathbf{E}\left(B_t B_s\right).$$

L'application $\mathbf{1}_{[0,t]} \longrightarrow B_t$ peut être étendue à une isométrie entre $\mathfrak{H}$ et l'espace gaussien associé au processus B. Dans ce qui suit, nous donnons les outils et propriétés fondamentales du calcul de Malliavin par rapport au processus B auxquelles nous ferons appel dans l'ensemble de cette thèse.

0.1.1 Opérateur de dérivation

Soit $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ la classe des fonctions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ infiniment dérivables et telles que f ainsi que toutes ses dérivées partielles soient bornées. Notons $\mathcal{S}$ la classe des fonctionnelles cylindriques de la forme

$$F = f(B(\varphi_1), \cdots, B(\varphi_n)) \tag{1}$$

où $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$, $\varphi_1, \cdots, \varphi_n \in \mathfrak{H}$ et $n \geq 1$ et où $B(\varphi)$ désigne l'image de φ par l'isométrie entre $\mathfrak{H}$ et l'espace gaussien associé au processus B. L'opérateur de dérivation, au sens de Malliavin, d'une fonctionnelle de la forme (1) est alors l'application $D : \mathcal{S} \longrightarrow L^2(\Omega; \mathfrak{H})$

définie par

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\varphi_1), \cdots, B(\varphi_n))\varphi_i.$$

Soient $k \geq 1$ et $p \geq 1$. On note $\mathbb{D}^{k,p}$ l'espace de Sobolev défini comme la fermeture de $\mathcal{S}$ par rapport à la norme

$$\|F\|_{k,p}^p = \mathbf{E}\left(|F|^p\right) + \sum_{j=1}^{k} \|D^j F\|_{L^p(\Omega;\mathfrak{H}^{\otimes j})}^p,$$

où D^j est l'opérateur de dérivation D itéré j fois.

0.1.2 Opérateur de divergence

L'opérateur de divergence (ou opérateur de Skorohod) noté δ est l'adjoint de l'opérateur D et est défini par la relation de dualité

$$\mathbf{E}\left(F\delta(u)\right) = \mathbf{E}\left(\langle DF, u\rangle_{\mathfrak{H}}\right), \quad u \in L^2\left(\Omega;\mathfrak{H}\right), F \in \mathbb{D}^{1,2}.$$

Le domaine de δ, noté $Dom(\delta)$, est l'ensemble des processus $u \in L^2\left(\Omega;\mathfrak{H}\right)$ tels que

$$\mathbf{E}\left(\langle DF, u\rangle_{\mathfrak{H}}\right) \leq C\sqrt{\mathbf{E}\left(F^2\right)},$$

pour tout $F \in \mathbb{D}^{1,2}$, et où C est une constante pouvant dépendre de u. De plus, si u est adapté à la filtration engendrée par B, l'opérateur de divergence coïncide avec l'intégrale d'Itô. C'est ce qui motive le fait que l'opérateur de divergence $\delta(u)$ puisse également être noté

$$\delta(u) = \int_0^T u_s \delta B_s.$$

0.1.3 Formule d'intégration par partie

Soient $F \in \mathbb{D}^{1,2}$ et $u \in Dom(\delta)$. Supposons de plus que les quantités aléatoires Fu et $F\delta(u) - \langle DF, u\rangle_{\mathfrak{H}}$ soient de carré intégrable. Alors on a la relation suivante, connue sous le nom de formule d'intégration par partie

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathfrak{H}}.$$

Soit (u_n) une suite de $Dom(\delta)$ qui converge vers u dans $L^2\left(\Omega;\mathfrak{H}\right)$. Supposons aussi que $\delta(u_n)$ converge dans $L^1\left(\Omega\right)$ vers une variable aléatoire de carré intégrable G. On a alors le résultat suivant,

$$u \in Dom(\delta) \text{ et } \delta(u) = G.$$

0.1.4 Intégrales stochastiques multiples

On suppose ici que B est un mouvement brownien, auquel cas $\mathfrak{H} = L^2([0,T])$. Soit $\mathcal{S}_n$ l'ensemble des fonctions simples à n variables de la forme

$$f = \sum_{i_1,\cdots,i_m=1}^{n} c_{i_1,\cdots,i_m} \mathbf{1}_{A_{i_1}} \cdots \mathbf{1}_{A_{i_m}}$$

où les coefficients $c_{i_1,\cdots,i_m}$ sont nuls si deux indices i_k et i_l sont égaux et où les boréliens $A_{i_k} \in \mathfrak{B}([0,T])$ sont deux à deux disjoints. Pour une telle fonction, on définit l'intégrale stochastique multiple d'ordre n par

$$I_n(f) = \sum_{i_1,\cdots,i_m=1}^{n} c_{i_1,\cdots,i_m} B\left(A_{i_1}\right)\cdots B\left(A_{i_m}\right)$$

où l'on a noté $B(A) := B\left(\mathbf{1}_A\right)$ pour $A \in \mathfrak{B}([0,T])$. On notera que pour tout $n \geq 1$, I_n est une application linéaire continue entre $\mathcal{S}_n$ et $L^2(\Omega)$. I_n vérifie la propriété suivante : pour tout $h \in \mathfrak{H}$ tel que $\|h\|_{\mathfrak{H}} = 1$, on a $I_n(h^{\otimes n}) = n!\mathbf{H}_n(B(h))$, où $\mathbf{H}_n(x)$ est le n$^{\text{ième}}$ polynôme de Hermite définit par

$$\mathbf{H}_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right),$$

pour tout $n \geq 1$ et avec $\mathbf{H}_0(x) = 1$. On définit alors le n$^{\text{ième}}$ chaos de Wiener, noté $\mathfrak{H}_n$, comme étant la fermeture dans $L^2(\Omega)$ du sous-espace vectoriel engendré par $\{\mathbf{H}_n(B(h));\ h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$. Par othogonalité des polynômes de Hermite, on a

$$\begin{cases} \mathbf{E}\left(I_n(f)I_m(g)\right) = n!\left\langle \tilde{f}, \tilde{g}\right\rangle_{L^2([0,T]^n)} \text{ si } m = n \\ \mathbf{E}\left(I_n(f)I_m(g)\right) = 0 \text{ si } m \neq n. \end{cases}$$

De plus, on a

$$I_n(f) = I_n\left(\tilde{f}\right)$$

où $\tilde{f}$ définie par $\tilde{f}(x_1,\cdots,x_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}([\![1,n]\!])} f(x_{\sigma(1)},\cdots,x_{\sigma(n)})$ désigne la fonction symétrisée de f. L'ensemble S_n étant dense dans $L^2([0,T]^n)$ pour tout $n \geq 1$, l'opérateur I_n peut être étendu à une application linéaire continue de $L^2([0,T]^n)$ dans $L^2(\Omega)$ et les propriétés énoncées ci-dessus sont également vérifiées par cette extension. Notons que I_n peut aussi s'écrire comme une intégrale stochastique itérée dans laquelle les intégrales s'entendent au sens d'Itô :

$$I_n(f) = n!\int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1,\cdots,t_n) dB_{t_1}\cdots dB_{t_n}.$$

Le produit de deux intégrales stochastiques multiples peut s'écrire comme une somme finie d'intégrales stochastiques multiples. En effet, si $f \in L^2([0,T]^n)$ et $g \in L^2([0,T]^m)$ sont des fonctions symétriques, alors

$$I_n(f)I_m(g) = \sum_{l=0}^{m\wedge n} l!\binom{m}{l}\binom{n}{l} I_{m+n-2l}\left(f \otimes_l g\right)$$

où la contraction $f \otimes_l g \in L^2([0,T]^{m+n-2l})$ est donnée, pour $1 \leq l \leq m \wedge n$ par

$$\begin{aligned}&\left(f \otimes_l g\right)\left(s_1,\cdots,s_{n-l},t_1,\cdots,t_{m-l}\right)\\ &= \int_{[0,T]^l} f(s_1,\cdots,s_{n-l},u_1,\cdots,u_l)g(t_1,\cdots,t_{m-l},u_1,\cdots,u_l)du_1\cdots du_l,\end{aligned}$$

le cas $l = 0$ correspondant au produit tensoriel. Toute variable aléatoire F de carré intégrable mesurable par rapport à la σ-algèbre engendrée par B peut être décomposée de manière unique comme une somme orthogonale de chaos de Wiener :

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{2}$$

où $f_0 = \mathbf{E}(F)$ et I_0 est l'application identité sur les constantes. Les noyaux f_n sont des fonctions symétriques déterminées de manière unique par la formule de Stroock

$$f_n = \frac{1}{n!}\mathbf{E}\left(D^n F\right).$$

0.1.5 Opérateur d'Ornstein-Uhlenbeck

Soit F définie par (2). L'opérateur d'Ornstein-Uhlenbeck, noté L, défini sur $Dom(L) = \mathbb{D}^{2,2}$, est donné par

$$LF = -\sum_{n\geq 0} nI_n(f_n).$$

Il existe une connection entre δ, D et L dans le sens où une variable aléatoire F appartient au domaine de L si et seulement si $F \in \mathbb{D}^{1,2}$ et $DF \in Dom(\delta)$. Dans ce cas, si de plus F est centrée, on a $\delta DF = -LF$.

0.2 Partie 1 : Modèles de régression et calcul de Malliavin

Dans la première partie de cette thèse, nous nous intéressons à des comportements asymptotiques de suites de variables aléatoires apparaissant de manière naturelle dans l'étude théorique de modèles de régression. Nous nous intéressons particulièrement à des modèles faisant intervenir le mouvement brownien fractionnaire en temps que processus d'erreur et en tant que régresseur. Avant d'aller plus loin, nous allons consacrer un avant propos au mouvement brownien fractionnaire et à ses propriétés.

0.2.1 Mouvement brownien fractionnaire

L'étude de phénomènes irréguliers a pris une place très importante dans beaucoup de domaines scientifiques, comme la mécanique des fluides, le traitement d'image ou encore les mathématiques financières. Le degré d'autosimilarité d'un processus est directement lié à sa régularité (höldérienne par exemple). Par ailleurs, l'utilisation de fonctions aléatoires est un outil pratique pour obtenir des modèles irréguliers. Le mouvement brownien fractionnaire se trouve à l'intersection de ces deux techniques. C'est dans [Kol40] que Kolmogorov introduit pour la première fois le mouvement brownien fractionnaire sous le nom de "spirales de Wiener" en le définissant comme étant l'unique processus gaussien centré $B^H = (B_t^H, t \geq 0)$ de fonction de covariance

$$R^H(s,t) = \mathbf{E}\left(B_s^H B_t^H\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right), \quad s,t \in \mathbb{R}_+,$$

où $H \in (0,1)$. C'est plus tard, lors de la publication des articles de Hurst [Hur51] et de Hurst, Black et Simaika [BHS65] consacrés à la capacité de stockage à long terme d'un réservoir, que le paramètre H prend le nom de "paramètre de Hurst". Le calcul stochastique par rapport au mouvement brownien fractionnaire débute avec le travail novateur de Mandelbrot et Van Ness [MVN68]. Ils donnent une représentation en moyenne mobile sur un intervalle infini de B^H basée sur le processus de Wiener $(W_t, t \geq 0)$,

$$B_t^H = \frac{1}{\Gamma(\frac{1}{2}+H)} \int_{-\infty}^{t} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) dW_s, \quad t \geq 0.$$

Ils baptisent ce processus "mouvement brownien fractionnaire". On remarquera que pour $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ est le mouvement brownien standard (ou processus de Wiener). Comme

$\mathbf{E}\left(|B_s^H - B_t^H|^2\right) = |s-t|^{2H}$, B^H admet une version continue (en utilisant le critère de continuité de Kolmogorov) dont les trajectoires ne sont presque sûrement hölderiennes que pour des exposants strictement inférieurs à H. Par conséquent, plus H est petit, plus les trajectoires sont irrégulières. Ce phénomène est du au fait que les accroissements, qui sont stationnaires pour toutes les valeurs de H, sont positivement corrélés dans le cas $H > \frac{1}{2}$ et négativement corrélés dans le cas $H < \frac{1}{2}$. Plus précisement, pour $H \neq \frac{1}{2}$ et $h > 0$ fixés,

$$\mathbf{E}\left(B_h^H\left(B_{t+h}^H - B_t^H\right)\right) \underset{t\to+\infty}{\sim} H(2H-1)h^2t^{2(H-1)}. \tag{3}$$

Une propriété simple mais souvent utile du mouvement brownien fractionnaire de paramètre de Hurst H est son auto-similarité : pour toute constante $a > 0$, les processus $(B_{at}^H, t \geq 0)$ et $(a^H B_t^H, t \geq 0)$ ont la même loi. Cette propriété a suscité un grand intérêt dans des domaines aussi variés que la modélisation d'actifs financiers, le traffic dans les réseaux de télécommunications ou encore les sciences naturelles (voir par exemple le livre de Mandelbrot [Man91]). Le mouvement brownien fractionnaire peut également être représenté sous forme intégrale sur l'intervalle compact $[0,T]$. Dans ce cas, on a (voir [DÜ99])

$$B_t^H = \int_0^t K_H(t,s)dW_s, \quad t \in [0,T],$$

où $K_H(t,s)$ est une fonction déterministe de t et de s. La propriété (3) implique que pour $H > \frac{1}{2}$ et pour tout $h > 0$ fixé, la série des corrélations diverge, i.e.

$$\sum_{n=1}^{\infty} |\mathbf{E}\left(B_h^H\left(B_{nh}^H - B_{(n-1)h}^H\right)\right)| = \infty.$$

Cette propriété est connue sous le nom de dépendance à long terme (ou longue mémoire). Cet aspect de longue mémoire est souvent considéré comme étant une motivation pour étudier les processus fractionnaires. Dans le cas $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, le mouvement brownien fractionnaire n'est ni un processus de Markov, ni une semimartingale relativement à sa filtration naturelle.

0.2.2 Chapitre 1 : Théorie asymptotique pour les modèles de régression fractionnaires par le calcul de Malliavin

En statistique, l'estimation par noyau est une méthode non-paramétrique d'estimation utilisée par exemple pour estimer des fonctions inconnues dans des modèles de régression. En économétrie, entre autre, la problématique de l'estimation d'une fonction f dans un modèle de régression général du type

$$y_i = f(x_i) + u_i, \quad i \geq 0$$

où $(u_i)_{i\geq 0}$ est l'incertitude et où $(x_i)_{i\geq 0}$ est le régresseur, revient souvent. L'estimateur à noyau usuel de $f(x)$, basé sur les observations $(y_i, x_i)_{i\geq 0}$, est donné par

$$\hat{f}(x) = \frac{\sum_{i=0}^n K_h(x_i - x)y_i}{\sum_{i=0}^n K_h(x_i - x)}$$

où K est un noyau strictement positif vérifiant $\int_{\mathbb{R}} K^2(y)dy = 1$ et $\int_{\mathbb{R}} yK(y)dy = 0$. En définissant $K_h(s) = \frac{1}{h}K(\frac{s}{n})$ où la “fenêtre” $h \equiv h_n$ vérifie $h_n \to 0$ quand $n \to \infty$, on sait que le comportement asymptotique de l'estimateur $\hat{f}$ est directement relié au comportement asymptotique de la suite

$$V_n = \sum_{i=1}^{n} K_h(x_i - x)u_i.$$

En effet, la différence $f - \hat{f}$ s'écrit sous la forme d'un quotient dont le numérateur est la suite V_n. L'étude de la limite en loi de la suite V_n est une problématique qui a suscité beaucoup d'interêt chez les statisticiens, les économètres, et plus généralement les mathématiciens. Cette limite a été étudiée dans le cas où x_t est une chaîne de Markov récurrente dans [KT01] et [KMT07], mais aussi dans le cas où x_t est une somme partielle de processus linéaires généraux dans [WP09a]. Un cas plus général a été traité dans [WP09b]. Pour plus de détails, ce sujet est également traité dans [PP01] et [Phi88]. Il est important de noter que tous les travaux mentionnés ci-dessus ont en commun une hypothèse sur le processus d'erreur $(u_i)_{i\geq 0}$: il est toujours considéré que ce processus est une suite d'accroissements d'une martingale.

Le Chapitre 1 de cette thèse est constitué de la publication [BT10] en collaboration avec Ciprian A. Tudor, paru dans *Journal of Theoretical Probability*.

L'objet de ces travaux est l'étude et la recherche, par le calcul stochastique par rapport au mouvement brownien fractionnaire et le calcul de Malliavin, de la limite de la suite V_n dans le cas où le processus d'erreur n'est plus une suite d'accroissements d'une martingale. Nous considérons comme noyau K le noyau gaussien, comme fenêtre $h_n = n^{-\alpha}$, comme régresseur un mouvement brownien fractionnaire B^{H_1} d'indice de Hurst $H_1 \in (0,1)$ et comme processus d'erreur les accroissements d'un mouvement brownien fractionnaire B^{H_2} d'indice de Hurst $H_2 \in (0,1)$ indépendant de B^{H_1}. La suite étudiée peut alors être réécrite sous la forme

$$S_n(x) = \sum_{i=0}^{n} K(n^\alpha(B_i^{H_1} - x)) \left(B_{i+1}^{H_2} - B_i^{H_2}\right).$$

Soit $x = 0$. La loi de $S_n \equiv S_n(0)$ conditionnellement à B^{H_1} est celle de

$$(a_n)^{\frac{1}{2}} Z, \tag{4}$$

où Z est une variable aléatoire gaussienne centrée réduite et où a_n est donné par

$$a_n = \sum_{i,j=0}^{n} K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\mathbf{E}\left((B_{i+1}^{H_2} - B_i^{H_2})(B_{j+1}^{H_2} - B_j^{H_2})\right).$$

Nous nous intérressons au cas où le terme diagonal, i.e. le cas où $i = j$, domine le terme non diagonal dans la convergence en norme L^2 de la suite S_n. Ce terme diagonal s'écrit

$$\langle S\rangle_n := \sum_{i=1}^{n} K^2(n^\alpha B_i^{H_1}). \tag{5}$$

Dans le cas $H = \frac{1}{2}$, $\langle S\rangle_n$ est réellement le crochet de la martingale discrète S_n, ce qui justifie cette notation entre crochet dans notre cas plus général. La condition requise pour que ce terme soit effectivement dominant est la suivante et porte sur α, H_1 et H_2.

$$\alpha - 4H_2 + H_1 + 2 > 0. \tag{6}$$

Le premier résultat que nous avons obtenu est le suivant et porte sur la limite en loi de $\langle S\rangle_n$.

Théorème 1. *La suite $\langle S\rangle_n$ définie par (5) et correctement renormalisée converge en loi vers le temps local $L^{H_1}(1,0)$ du mouvement brownien fractionnaire B^{H_1} pondéré par le carré de la norme $L^2(\mathbb{R})$ du noyau gaussien :*

$$n^{\alpha+H_1-1}\langle S\rangle_n \to \|K\|^2_{L^2(\mathbb{R})}L^{H_1}(1,0).$$

A cause de (4), ce résultat est central dans la preuve du théorème suivant ayant pour objet la convergence en loi de la suite S_n .

Théorème 2. *Sous l'hypothèse*

$$\alpha < 1 - H_1, \tag{7}$$

la suite (S_n) correctement renormalisée converge en loi vers un mouvement brownien standard W changé de temps par le temps local du mouvement brownien fractionnaire B^{H_1} et pondéré par le carré de la norme $L^2(\mathbb{R})$ du noyau gaussien :

$$n^{\alpha+H_1-1}S_n \underset{n\to+\infty}{\longrightarrow} \|K\|^2_{L^2(\mathbb{R})}W_{L^{H_1}(1,0)}.$$

De plus, W et B^{H_1} sont indépendants.

Ce théorème est une extension naturelle d'autres résultats obtenus dans le cas $H = \frac{1}{2}$ (voir [WP09a] et [WP09b]). Dans ces travaux, c'est le temps local du mouvement brownien standard qui apparaît. La condition (7) n'apparaît pas dans le cas $H = \frac{1}{2}$ et est du ici au fait que l'on utilise également un mouvement brownien fractionnaire en tant que régresseur. Nous avons ensuite étendu ce résultat au cas de la convergence stable, qui est plus forte que la convergence en loi.

Théorème 3. *Soit $(G_t)_{t\geq 0}$ un processus stochastique indépendant de B^{H_1} et adapté à la filtration de B^{H_2} tel que pour tout $t \geq 0$ la variable aléatoire G_t appartienne à $\mathbb{D}^{1,2}$ et $\|D_sG_t\| \leq C$ pour tous s, t et ω. Alors, sous les hypothèses (6) et (7), le vecteur $(S_n, (G_t)_{t\geq 0})$ converge vers le vecteur $\left(cW_{L^{H_1}(1,0)}, (G_t)_{t\geq 0}\right)$ au sens des lois fini-dimensionnelles, où c est une constante positive.*

La preuve de ce théorème fait intensivement appel au calcul stochastique par rapport au mouvement brownien fractionnaire. On y utilise par exemple une formule d'Itô pour des intégrales stochastiques par rapport au mouvement brownien fractionnaire ainsi que le calcul de Malliavin et plus particulièrement l'opérateur de divergence fractionnaire.

0.3 Partie 2 : Bornes d'erreurs dans les TCL par la méthode de Stein et le calcul de Malliavin

La deuxième partie de cette thèse est consacrée à des travaux portant sur l'application de la méthode de Stein et du calcul de Malliavin à l'obtention de bornes de type Berry-Esséen pour des théorèmes de la limite centrale ainsi que sur d'autres applications, notamment à la preuve de théorèmes de type Cramér pour la loi Gamma sur l'espace de Wiener. Avant toute chose, nous commençons par introduire la méthode de Stein combinée au calcul de Malliavin compte tenu du fait que ces techniques jouent un rôle central et prépondérant dans cette partie.

0.3.1 Méthode de Stein et calcul de Malliavin

Soit $(X_i; i = 1, \cdots, n)$ une suite i.i.d de variables aléatoires telles que $\mathbf{E}(X_i) = 0$ et $\mathbf{E}(X_i^2) = \sigma^2$ et considérons $W_n = \sum_{i=1}^n X_i$. Il n'est en général pas possible de trouver la loi exacte de W_n, ou si oui, elle peut être difficile à manipuler. D'un autre coté, le théorème de la limite centrale nous dit que

$$\mathbf{P}\left(\frac{W_n}{n} \leq t\right) \underset{n\to+\infty}{\longrightarrow} \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx, \quad \forall t \in \mathbb{R}.$$

La question qui se pose est alors celle de la qualité d'une telle approximation. La réponse à cette question est donnée par la méthode de Stein : en 1972, Stein introduit une méthode pour borner la distance entre la loi d'une variable aléatoire X et la gaussienne centrée réduite $\mathcal{N}(0,1)$ [Ste72]. En 1975, Chen étend les travaux de Stein à la loi de Poisson [Che75]. Depuis, la méthode a été généralisée à beaucoup d'autres distributions : la loi uniforme en 1989 [DZ91], la loi Binomiale en 1991 [Ehm91], la loi de Poisson composée en 1992 [BCL92], la loi multinomiale en 1992 [Loh92], la loi Gamma en 1994 [Luk94] ou encore la loi géométrique en 1995 [Pek96]. Pour plus de détails, nous conseillons la très bonne revue des résultats de ce type par Reinert [Rei05].

Opérateurs caractéristiques

Soit Z une variable aléatoire telle que $\mathcal{L}(Z) = \mu$ où la notation $\mathcal{L}(Z)$ désigne la loi de Z. Un opérateur caractéristique de μ est un opérateur A_μ sur une certaine classe de fonctions $\mathcal{F}$, tel que, pour toute variable aléatoire X,

$$\forall f \in \mathcal{F}, \quad \mathbf{E}[A_\mu f(X)] = 0 \Longleftrightarrow \mathcal{L}(X) = \mathcal{L}(Z) = \mu.$$

Il existe en général une infinité de tels opérateurs pour une même loi. La question duquel vaut il mieux choisir est toujours une question ouverte. Nous donnons quelques exemples de ces opérateurs caractéristiques dans le cas de lois usuelles. Ces résultats sont connus sous le nom de *Lemme de Stein* pour la loi concernée.

- ▷ Loi Normale $\mathcal{N}(0,1)$: $A_\mu f(X) = f'(X) - Xf(X)$ pour $f : \mathbb{R} \to \mathbb{R}$.
- ▷ Loi Gamma $\Gamma(\alpha, 1)$: $A_\mu f(X) = Xf'(X) - (X - (\alpha + 1)))f(X)$ pour $f : \mathbb{R}_+ \to \mathbb{R}$.
- ▷ Loi de Poisson $\mathcal{P}(\lambda)$: $A_\mu f(X) = \lambda f(X+1) - Xf(X)$ pour $f : \mathbb{N} \to \mathbb{N}$.

Principe de la méthode de Stein

Soit W une variable aléatoire qui nous intéresse ($W = \sum_{i=1}^n X_i$ avec $(X_i; i = 1, \cdots, n)$ une suite i.i.d de variables aléatoires par exemple) dont on ne connait pas la loi. Soit Z une variable aléatoire d'opérateur caractéristique A_μ que l'on suppose être une bonne approximation de W. L'idée de la méthode de Stein est d'essayer d'évaluer la taille (ou de la borner si une mesure directe s'avérait impossible) de $\mathbf{E}[A_\mu f(W)]$ pour toute fonction $f \in \mathcal{F}$. Si cette taille s'avérait être petite, cela impliquerait que W est proche de Z. La taille de $\mathbf{E}[A_\mu f(W)]$ détermine la distance entre W et Z. Si de plus W dépend d'un paramètre n tendant vers l'infini, alors $\mathbf{E}[A_\mu f(W)]$ sera la vitesse de convergence de la suite W_n vers Z. La propriété que nous venons de décrire est une consequence du lemme de Stein.

L'équation de Stein

Pour mettre en pratique cette idée, il faut mettre en relation l'opérateur caractéristique de la variable aléatoire approximante Z et la quantité $h(x) - \mathbf{E}(h(Z))$. En effet, beaucoup de distances usuelles peuvent être exprimées sous la forme

$$d(X,Y) = \sup_{h\in\mathcal{H}} |\mathbf{E}(h(X)) - \mathbf{E}(h(Y))|, \tag{8}$$

où $\mathcal{H}$ est une classe de fonctions test appropriée. Ci-dessous, on trouvera des exemples de différentes distances pouvant s'exprimer ainsi.

- ▷ Si $\mathcal{H} = \{h : h = \mathbf{1}_A, A \in \mathfrak{B}\}$, alors nous obtenons la distance en variation totale.
- ▷ Si $\mathcal{H} = \{h : \|h\|_\infty \leq 1\}$, alors nous obtenons la distance de Wasserstein.
- ▷ Si $\mathcal{H} = \left\{h : h = \mathbf{1}_{(-\infty,a]}, a \in \mathbb{R}\right\}$, alors nous obtenons la distance de Kolmogorov.

Cette relation est établie par l'équation de Stein : soit Z une variable aléatoire telle que $\mathcal{L}(Z) = \mu$. Pour une fonction donnée $h \in \mathcal{H}$, résoudre l'équation de Stein, c'est trouver une fonction f_h telle que

$$A_\mu f_h(x) = h(x) - \mathbf{E}(h(Z)).$$

Cette fonction f_h déterminée, on a

$$|\mathbf{E}\left[A_\mu f_h(W)\right]| = |\mathbf{E}(h(W)) - \mathbf{E}(h(Z))|.$$

On voit donc que si $|\mathbf{E}\left[A_\mu f_h(W)\right]|$ est petit, alors $|\mathbf{E}(h(W)) - \mathbf{E}(h(Z))|$ aussi et inversement.

Réduction du problème

Au moyen de l'équation et du lemme de Stein, le problème qu'est l'estimation de la distance entre W et Z est réduit à la mesure de la taille de $\mathbf{E}\left[A_\mu f_h(W)\right]$, c'est à dire

$$d(W,Z) = \sup_{h\in\mathcal{H}} |\mathbf{E}(h(W)) - \mathbf{E}(h(Z))| = \sup_{h\in\mathcal{H}} |\mathbf{E}\left[A_\mu f_h(W)\right]|.$$

Il ne reste plus qu'à étudier de manière plus approfondie l'équation de Stein afin de mettre en lumière certaines propriétés de sa solution f_h.

Propriétés des solutions de l'équation de Stein

Soit h une fonction bornée de $\mathbb{R}$ dans $\mathbb{R}$. Pour tout h, il existe une fonction absolument continue f_h solution de l'équation de Stein associée à la loi normale,

$$h(x) - \mathbf{E}(h(Z)) = f'(x) - xf(x)$$

pour tout x et vérifiant $\|f_h\|_\infty \leq \sqrt{\frac{\pi}{2}}\|h - \mathbf{E}(h(Z))\|_\infty$ et $\|f_h'\|_\infty \leq 2\|h - \mathbf{E}(h(Z))\|_\infty$ où $Z \sim \mathcal{N}(0,1)$. Si h est lipschitzienne mais pas forcément bornée, alors $\|f_h\|_\infty \leq \|h\|_\infty$, $\|f_h'\|_\infty \leq \|h'\|_\infty$ et $\|f_h''\|_\infty \leq 2\|h''\|_\infty$. On peut également obtenir ce genre de résultats dans les autres cas, ils ne sont pas limités au cas gaussien.

Liens avec le calcul de Malliavin

Nous exposons ici le cas de l'approximation gaussienne, mais les résultats établis ici sont parfaitement transposables à d'autres types d'approximation (loi Gamma, loi de Poisson, mais aussi la loi de Pearson [EV10] ou encore les mesures invariantes d'une diffusion [KT11]). Nous avons le théorème suivant, établit par Nourdin et Peccati dans [NP09c].

Théorème 4. *Soit $F \in \mathbb{D}^{1,2}$ telle que $\mathbf{E}(F) = 0$ et soit $\psi_F = \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}$. Z désigne une variable aléatoire gaussienne centrée réduite. On a alors*

$$d_W(F,Z) \leq \mathbf{E}\left(|1-\psi_F|\right) \leq \mathbf{E}\left(|1-\psi_F|^2\right)^{\frac{1}{2}},$$

où d_W désigne la distance de Wasserstein qui n'est prise ici qu'à titre d'exemple à des fins de simplicité d'écriture. Ce résultat est valable également pour beaucoup d'autres distances de la forme (8).

Preuve : On a

$$d_W(F,Z) = \sup_{h \in \mathcal{H}_W} |\mathbf{E}(h(F)) - \mathbf{E}(h(Z))|.$$

En se servant de la solution de l'équation de Stein, on peut écrire

$$d_W(F,Z) = \sup_{h \in \mathcal{H}_W} |\mathbf{E}\left[f_h'(F) - f_h(F)F\right]|.$$

D'autre part, comme $\mathbf{E}(F) = 0$, on a $F = LL^{-1}F = -\delta DL^{-1}F = \delta(-DL^{-1}F)$. En appliquant la formule d'intégration par partie, on obtient

$$\mathbf{E}\left(f_h(F)F\right) = \mathbf{E}\left(f_h(F)\delta(-DL^{-1}F)\right) \overset{\text{IPP}}{=} \mathbf{E}\left(\left\langle Df_h(F), -DL^{-1}F\right\rangle_{\mathfrak{H}}\right).$$

On obtient finalement, après simplification du terme $Df_h(F)$,

$$\mathbf{E}\left(f_h(F)F\right) = \mathbf{E}\left(f_h'(F)\left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}\right).$$

D'où

$$\begin{aligned} d_W(F,Z) &= \sup_{h \in \mathcal{H}_W} |\mathbf{E}\left[f_h'(F) - f_h'(F)\left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}\right]| \\ &\leq \sup_{h \in \mathcal{H}_W} \|f_h'\|_\infty \mathbf{E}\left[|1 - \left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}|\right] \\ &\leq \underbrace{\sup_{h \in \mathcal{H}_W} \|h'\|_\infty}_{=1} \mathbf{E}\left[\left(1 - \left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}\right)^2\right]^{\frac{1}{2}}. \end{aligned}$$

■

Mesurer la distance entre une variable aléatoire de $\mathbb{D}^{1,2}$ et une gaussienne centrée réduite revient donc à évaluer le terme $\mathbf{E}\left[\left(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\right)^2\right]$.

Théorèmes limites sur les chaos de Wiener

Le théorème suivant, connu sous le nom de "Fourth moment Theorem", a été démontré par Nualart et Peccati dans [NP05] et complété par la suite dans [NOL08] et [NP09c]. En voici l'énoncé.

Théorème 5. *Soit $q \geq 2$ et soit $F_n = I_n(f_n)$, $n \geq 1$, une suite du $q^{\text{ième}}$ chaos telle que $\mathbf{E}(F_n^2) \underset{n \to +\infty}{\to} 1$. Alors les propriétés suivantes sont équivalentes :*

1. *$F_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) = \mathcal{L}(Z)$.*

2. $d(F_n, Z) \underset{n\to+\infty}{\to} 0$.
3. *Pour tout* $p = 1, \cdots, q-1$, $\|f_n \tilde{\otimes}_p f_n\|^2_{\mathfrak{H}^{\odot 2(p-q)}} \underset{n\to+\infty}{\to} 0$.
4. $Var\left[\frac{1}{q}\|DF_n\|^2_{\mathfrak{H}}\right] \underset{n\to+\infty}{\to} 0$.
5. $\mathfrak{K}_4(F_n) = \mathbf{E}\left(F_n^4\right) - 3\mathbf{E}\left(F_n^2\right)^2 \underset{n\to+\infty}{\to} 0$.

Les équivalences entre *1.*, *3.* et *5.* ont été démontrées par Nualart et Peccati dans [NP05] au moyen du calcul chaotique et du théorème de Dambis, Dubins-Schwarz. L'équivalence entre *1.* et *4.* a été démontrée par Nualart et Ortiz-Latorre dans [NOL08] en utilisant une approche basée sur l'équation différentielle satisfaite par la fonction caractéristique de la gaussienne centrée réduite. L'équivalence entre *2.* et *4.* a été démontrée par Nourdin et Peccati dans [NP09c], créant ainsi le lien entre méthode de Stein et calcul de Malliavin. On notera que le fait que *1.* entraine *2.* est remarquable compte tenu du fait que les topologies induites par les distances que l'on considère sont plus fortes que la topologie de la convergence faible. L'implication *1.* $\to$ *5.* traduit le fait qu'il suffit de vérifier que $\mathbf{E}\left(F_n^4\right) \underset{n\to+\infty}{\to} 3$ et $\mathbf{E}\left(F_n^2\right) \underset{n\to+\infty}{\to} 1$ pour conclure. C'est ce qui a donné son nom au théorème. Une version multidimensionnelle de ce résultat a été démontrée par Peccati et Tudor dans [PT05].

0.3.2 Chapitre 2 : Bornes de Berry-Esséen pour les moyennes mobiles à mémoire longue par la méthode de Stein et le calcul de Malliavin

La combinaison récente de la méthode de Stein et du calcul de Malliavin par Nourdin et Peccati [NP09c] (on pourra se reporter également à [NP09b] et [NP10]) a fourni une palette d'outils permettant de mesurer la distance entre la loi d'une variable aléatoire d'un chaos de Wiener et la loi normale centrée réduite. Cette mesure d'éloignement de lois a permis de donner des bornes d'erreurs ainsi que d'estimer la vitesse de convergence dans de nombreux théorèmes de la limite centrale. Cela permet d'avoir une vision bien plus fine du comportement asymptotique d'une suite de variables aléatoires convergeant vers la loi normale centrée réduite (ou vers d'autres lois, comme la loi Gamma ou la loi de Poisson). Ces problématiques ont été beaucoup étudiées récemment, notamment dans [NP09c], où les auteurs donnent des bornes de Berry-Esséen pour le théorème de la limite centrale pour des fonctionnelles du mouvement brownien fractionnaire ainsi que dans [NP09b] où ces techniques sont appliquées à des théorèmes de la limite centrale pour des fonctionnelles quadratiques de Toeplitz de processus stationnaires continus. On pourra également se reporter à [Led11] qui reprend le théorème du moment d'ordre quatre de Nualart et Peccati pour l'étendre aux chaos d'un processus de Markov au lieu des chaos associés à un processus gaussien.

Le Chapitre 2 de cette thèse est constitué de la publication [BT11a] en collaboration avec Ciprian A. Tudor, paru dans *Stochastic Analysis and Applications*.

Cet article montre comment il est possible de mettre en oeuvre les techniques décrites ci-dessus pour calculer de manière explicite des bornes de Berry-Esséen dans les théorèmes de la limite centrale pour des processus moyenne-mobile à mémoire longue. Une application intérressante de ces résultats est la preuve du théorème de Hsu-Robbins et du théorème de Spitzer pour ces moyennes mobiles à mémoire longue. Plus précisement, ces dernières sont définies par

$$X_n = \sum_{i\geq 1} a_i \varepsilon_{n-i}, n \in \mathbb{Z}$$

où les innovations ε_i sont des variables aléatoires centrées indépendantes et identiquement distribuées dont au moins les moments d'ordre 2 sont finis et où les moyennes mobiles a_i sont de la forme $a_i = i^{-\beta}L(i)$ avec $\beta \in (\frac{1}{2}, 1)$ et L une fonction à variations lentes vers l'infini. La fonction de covariance $\rho(m) = \mathbf{E}\left(X_0 X_m\right)$ se comporte comme $c_\beta m^{-2\beta+1}$ quand $m \to \infty$ et n'est donc pas sommable étant donné que $\beta < 1$. C'est pour cette raison que X_n est généralement appellé moyenne mobile à mémoire longue. Soit K une fonction déterministe dont le rang de Hermite vaut q et satisfaisant $\mathbf{E}(K^2(X_n)) < \infty$ et soit S_N la suite définie par

$$S_N = \sum_{n=1}^{N} \left[K(X_n) - \mathbf{E}\left(K(X_n)\right)\right].$$

Il a été prouvé dans [HH97] (voir aussi [Wu06]) que pour des constantes $c_1(\beta, q)$ et $c_2(\beta, q)$ positives et ne dépendant que de q et β :

i. Si $q > \frac{1}{2\beta-1}$, alors la suite $c_1(\beta, q)\frac{1}{\sqrt{N}} S_N$ converge en loi vers une loi normale centrée réduite.

ii. Si $q < \frac{1}{2\beta-1}$, alors la suite $c_2(\beta, q) N^{\beta q - \frac{q}{2} - 1} S_N$ converge en loi vers une variable aléatoire de Hermite d'ordre q.

Ce dernier point est une variante du théorème de la limite non centrale de [DM79] et [Taq79].

Remarque 1. *La variable aléatoire de Hermite $Z^{(q)}$ est définie comme étant la valeur au temps 1 du processus de Hermite d'ordre q et d'indice d'autosimilarité $\frac{q}{2} - q\beta + 1$ défini dans [CTV09].*

Afin d'être en mesure d'appliquer les techniques décrites plus haut pour mesurer l'éloignement entre S_n et sa loi limite, nous considérons le cas où les innovations ε_i sont les accroissements du mouvement brownien W sur $\mathbb{R}$, où la fonction K est un polynôme de Hermite d'ordre q et où la fonction L est identiquement égale à 1. Dans ce cadre d'hypothèses, la variable aléatoire X_n est une intégrale de Wiener par rapport à W, et $H_q(X_n)$ peut être également écrit comme une intégrale multiple de Wiener-Itô d'ordre q par rapport à W. Nous avons tout d'abord traité le cas du théorème de la limite centrale pour lequel nous avons obtenu le résultat suivant.

Théorème 6. *Soit $Z_N = \frac{1}{\sigma\sqrt{N}} S_N$ avec $\sigma = q! \sum_{m=-\infty}^{+\infty} \left(\sum_{i=1}^{+\infty} a_i a_{i+|m|}\right)^q$. Sous l'hypothèse que $q > (2\beta - 1)^{-1}$, Z_N converge en loi vers $Z \sim \mathcal{N}(0, 1)$. De plus, il existe une constante C_β, ne dépendant que de β, telle que, pour tout $N \geq 1$,*

$$\sup_{z\in\mathbb{R}} \left|\mathbf{P}(Z_N \leq z) - \mathbf{P}(Z \leq z)\right| \leq C_\beta \begin{cases} N^{\frac{q}{2}+\frac{1}{2}-q\beta} & si \quad \beta \in \left(\frac{1}{2}, \frac{q}{2q-2}\right] \\ N^{\frac{1}{2}-\beta} & si \quad \beta \in \left[\frac{q}{2q-2}, 1\right) \end{cases}$$

Remarque 2. *Il est bon de noter que ce résultat reste valable pour d'autres distances comme la distance de Wasserstein ou la distance en variation totale.*

Remarque 3. *Nous attirons l'attention sur le fait que les résultats du théorème précédent sont cohérent avec ceux du Théorème 4.1 dans [NP09c]. En effet, dans [NP09c] une suite $Y_n = B^H_{n+1} - B^H_n$ est considérée à la place de X_n, où B^H est un mouvement brownien* *fractionnaire. Ces résultats peuvent être comparés car les fonctions de covariance considérées sont les mêmes. Dans [NP09c], la fonction de covariance $\rho'(m) = \mathbf{E}(Y_0 Y_m)$ de Y se comporte asymptotiquement comme m^{2H-2} alors qu'ici, la covariance de X se comporte asymptotiquement comme $m^{-2\beta+1}$. β correspond donc à $\frac{3}{2} - H$.*

Dans le cas du théorème de la limite non centrale, nous avons obtenu le résultat suivant pour la distance en variation totale.

Théorème 7. *Sous l'hypothèse $q < \frac{1}{2\beta-1}$, $h_{k,\beta}^{-1} N^{\beta k - \frac{k}{2} - 1} S_N$ converge en loi vers $Z^{(q)}$, une variable aléatoire de Hermite d'ordre q. De plus, il existe une constante positive $C_0(q,\beta)$, ne dépendant que de q et de β, telle que, pour tout $N \geq 1$,*

$$d_{TV}\left(h_{q,\beta}^{-1} N^{\beta q - \frac{q}{2} - 1} S_N, Z^{(q)}\right) \leq C_0(q,\beta) N^{2\beta q - q - 1}$$

où

$$h_{q,\beta}^2 = \frac{2c_\beta^q}{q!(-2\beta q + q + 1)(-2\beta + q + 2)}$$

et où c_β est définie par $c_\beta = \boldsymbol{\beta}(2\beta - 1, 1 - \beta)$ et $\boldsymbol{\beta}$ la fonction Beta.

Fort de ces deux théorèmes, nous donnons une application de ces derniers à la preuve du théorème de Hsu-Robbins et du théorème de Spitzer pour les moyennes mobiles à mémoire longue. Ces théorèmes fournissent un autre moyen de comparer les lois de S_n et de sa limite. Plus précisement, le but du théorème de Spitzer est de determiner le comportement asymptotique des suites $f_1(\varepsilon)$ et $f_2(\varepsilon)$ définies ci-dessous quand $\varepsilon \to 0$.

$$f_1(\varepsilon) = \sum_{N \geq 1} \frac{1}{N} \mathbf{P}\left(|S_N| > \varepsilon N\right).$$

quand $q > \frac{1}{2\beta-1}$ et

$$f_2(\varepsilon) = \sum_{N \geq 1} \frac{1}{N} \mathbf{P}\left(|S_N| > \varepsilon N^{-2\beta q + q + 2}\right).$$

quand $q < \frac{1}{2\beta-1}$.

Remarque 4. *On notera que $f_1(\varepsilon)$ peut être décomposée de la manière suivante*

$$\begin{aligned} f_1(\varepsilon) &= \sum_{N \geq 1} \frac{1}{N} P\left(\sigma^{-1} \frac{1}{\sqrt{N}} |S_N| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) \\ &= \sum_{N \geq 1} \frac{1}{N} P\left(|Z| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) \\ &\quad + \sum_{N \geq 1} \frac{1}{N} \left[P\left(\sigma^{-1} \frac{1}{\sqrt{N}} |S_N| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) - P\left(|Z| > \frac{\varepsilon\sqrt{N}}{\sigma}\right)\right] \end{aligned}$$

Le deuxième terme peut alors être évalué par la méthode de Stein car il correspond justement à la distance de Kolmogorov entre S_n et sa limite.

Nous avons prouvé le résultat suivant en utilisant cette décomposition et la méthode de Stein.

Proposition 1. *Lorsque* $q > \frac{1}{2\beta-1}$, *on a*

$$\lim_{\varepsilon\to 0} \frac{1}{-\log(\varepsilon)} f_1(\varepsilon) = 2$$

et lorsque $q < \frac{1}{2\beta-1}$ *on a*

$$\lim_{\varepsilon\to 0} \frac{1}{-\log(\varepsilon)} f_2(\varepsilon) = \frac{1}{1+\frac{q}{2}-\beta q}.$$

Dans le même esprit, le théorème de Hsu-Robbins pour les moyennes mobiles à mémoire longue permet de determiner le comportement asymptotique des suites $g_1(\varepsilon)$ et $g_2(\varepsilon)$ définies ci-dessous quand $\varepsilon \to 0$.

$$g_1(\varepsilon) = \sum_{N\geq 1} \mathbf{P}\left(|S_N| > \varepsilon N\right)$$

lorsque $q > \frac{1}{2\beta-1}$ et

$$g_2(\varepsilon) = \sum_{N\geq 1} \mathbf{P}\left(|S_N| > \varepsilon N^{-2\beta q+q+2}\right)$$

lorsque $q < \frac{1}{2\beta-1}$. Nous avons prouvé le résultat suivant (de manière similaire au cas de Spitzer).

Proposition 2. *Lorsque* $q > \frac{1}{2\beta-1}$, *on a*

$$\lim_{\varepsilon\to 0} \sigma^{-1}\varepsilon^2 g_1(\varepsilon) = 1$$

et lorsque $q < \frac{1}{2\beta-1}$, *on a*

$$\lim_{\varepsilon\to 0} (h_{q\beta}^{-1}\varepsilon)^{\frac{1}{1+\frac{q}{2}-\beta q}} g_2(\varepsilon) = \mathbf{E}\left|Z^{(q)}\right|^{\frac{1}{1+\frac{q}{2}-\beta q}}.$$

Remarque 5. *Quand* $\beta = 1$ *dans ces deux propositions, on retrouve bien les résultats classiques. Ces deux propositions sont donc cohérentes avec la littérature existante à ce sujet dans des cas plus particuliers.*

0.3.3 Chapitre 3 : Théorèmes de Cramér pour la loi Gamma

Le théorème de Cramér [Cra36] stipule que deux variables aléatoires réelles X et Y indépendantes dont la somme $X+Y$ est gaussienne sont elles mêmes nécessairement gaussiennes. Par récurrence, cela implique que tous les éléments d'une somme finie de variables aléatoires suivant une loi normale sont eux mêmes gaussiens. Une version asymptotique de ce théorème a été démontrée par C.A. Tudor sur l'espace de Wiener [Tud11].

Le Chapitre 3 de cette thèse est constitué de la publication [BT11b] en collaboration avec Ciprian A. Tudor, paru dans *Electronic Communications in Probability.*

Le but de ce chapitre est de démontrer le même résultat que celui de Cramér pour la loi Gamma. En effet, il est bien connu que si $X \sim \Gamma(a,\lambda)$ et $Y \sim \Gamma(b,\lambda)$ avec $a, b, \lambda > 0$ et que de plus X est indépendante de Y, alors la somme $X+Y$ suit la loi $\Gamma(a+b,\lambda)$. Nous avons montré que l'implication contraire est vraie uniquement sur l'espace de Wiener, i.e. pour des variables aléatoires appartenant à un chaos d'ordre fixe. Nous avons également montré, par un contre-exemple bien choisi, pourquoi cette même propriété n'est pas vérifiée en dehors de ce cadre d'hypothèses. Nous avons également démontré une version asymptotique de ce résultat. Les preuves de ces résultats sont basées sur la méthode de Stein et le calcul de Malliavin, et plus spécifiquement sur une caractérisation de la loi Gamma en terme d'opérateurs différentiels du calcul de Malliavin donnée dans [NP09c]. Nous précisons maintenant le cadre de notre étude. Nous travaillons (pour des questions de simplicité d'écriture, mais sans perte de généralité) avec la loi Gamma dite centrée $F(\nu)$. Une variable aléatoire est dite Gamma centrée si elle est de la forme

$$F(\nu) \overset{\text{Law}}{=} 2G(\nu/2) - \nu, \quad \nu > 0,$$

où $G(\nu/2) := F(\nu/2, 1)$ suit une loi Gamma de paramètres $\nu/2$ et 1. Cela implique que $G(\nu/2, 1)$ est une variable aléatoire positive p.s de densité $g(x) = \frac{x^{\frac{\nu}{2}-1}\mathrm{e}^{-x}}{\Gamma(\nu/2)}\mathbf{1}_{(0,\infty)}(x)$. La fonction caractéristique de la loi $F(\nu)$ est donnée par

$$\mathbf{E}\left(e^{i\lambda F(\nu)}\right) = \left(\frac{e^{-i\lambda}}{\sqrt{1-2i\lambda}}\right)^{\nu}, \qquad \lambda \in \mathbb{R}.$$

Notre premier résultat est le théorème de Cramér pour la loi Gamma sur l'espace de Wiener.

Théorème 8. *Soit $Z = X + Y = I_{q_1}(f) + I_{q_2}(h)$, $q_1, q_2 \geq 2$ avec $f \in L^2(T^{q_1})$ et $h \in L^2(T^{q_2})$ des fonctions symétriques, telle que X et Y soient indépendantes et telle que*

$$\mathbf{E}\left(Z^2\right) = 2\nu, \quad \mathbf{E}\left(X^2\right) = q_1!\,\|f\|^2_{L^2(T^{q_1})} = 2\nu_1, \quad \mathbf{E}\left(Y^2\right) = q_2!\,\|h\|^2_{L^2(T^{q_2})} = 2\nu_2$$

avec $\nu = \nu_1 + \nu_2$. De plus, supposons que $Z \sim F(\nu)$. Alors,

$$X \sim F(\nu_1) \quad \textit{et} \quad Y \sim F(\nu_2).$$

Remarque 6. *Le cas $q_1 = q_2$ est immédiat à partir des résultats de Nourdin et Peccati contenus dans [NP09a]. En revanche, le cas $q_1 \neq q_2$ est différent et ne peut pas être traité directement à partir des résultats de [NP09a].*

Nous avons également démontré une version asymptotique de ce résultat.

Théorème 9. *Soit $Z_k = X_k + Y_k = I_{q_1}(f_k) + I_{q_2}(h_k)$ avec $f_k \in L^2(T^{q_1})$ et $h_k \in L^2(T^{q_2})$ des fonctions symétriques. On suppose que $k \geq 1$, $q_1, q_2 \geq 2$ et que X_k et Y_k sont indépendantes pour tout $k \geq 1$. De plus, on suppose également que*

$$\mathbf{E}\left(Z_k^2\right) \underset{k\to+\infty}{\to} 2\nu, \quad \mathbf{E}\left(X_k^2\right) = q_1!\,\|f\|^2_{L^2(T^{q_1})} \underset{k\to+\infty}{\to} 2\nu_1, \quad \mathbf{E}\left(Y_k^2\right) = q_2!\,\|h\|^2_{L^2(T^{q_2})} \underset{k\to+\infty}{\to} 2\nu_2$$

avec $\nu = \nu_1 + \nu_2$. Pour finir, supposons que $Z_k \underset{k\to+\infty}{\to} F(\nu)$ en loi. Alors,

$$X_k \underset{k\to+\infty}{\longrightarrow} F(\nu_1) \quad \textit{et} \quad Y_k \underset{k\to+\infty}{\longrightarrow} F(\nu_2).$$

On peut faire la remarque suivante à propos de ces deux théorèmes et de leurs implications.

Remarque 7. *Le Corrolaire 4.4. dans [NP09b] montre qu'il ne peut pas exister de variable aléatoire suivant une loi Gamma dans un chaos d'ordre supérieur ou égal à quatre (Nous conjecturons même que de telles variables aléatoires sont forcément dans le chaos d'ordre 2). De ce point de vue, le théorème asymptotique est plus intérressant car il s'applique à une bien plus grande variété de variables aléatoires que le théorème non asymptotique. En effet, il existe une classe très grande de variable aléatoire suivant asymptotiquement une loi Gamma.*

Nous fournissons également un contre-exemple démontrant que la propriété de Cramér n'est plus vraie en dehors des chaos de Wiener.

0.3.4 Chapitre 4 : Sommes autonormalisées et calcul de Malliavin

L'étude des sommes de variables aléatoires autonormalisées a commencé historiquement suite aux travaux de William Gosset [Stu08], mieux connu sous son nom de plume, Student. Il considère alors le problème de l'inférence statistique, à partir d'un échantillon $X_1, \cdots, X_n$ d'observations i.i.d, de la moyenne μ lorsque l'écart type σ de la distribution sous-jacente est inconnu. En considérant la moyenne empirique $X_n = \frac{1}{n}\sum_{i=1}^n X_i$ ainsi que la variance empirique $s_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$, il trouve la loi de la statistique $T_n = \sqrt{n}\frac{\bar{X}_n - \mu}{s_n}$ pour des variables aléatoires X_i gaussiennes. Quand le test de Student pour $\mu = \mu_0$ fut introduit, cela donna lieu à l'étude de statistiques du type, quand $\mu_0 = 0$ par exemple,

$$T_n = \frac{\sqrt{n}\bar{X}_n}{s_n} = \frac{S_n}{V_n}\left[\frac{n-1}{n-(S_n/V_n)^2}\right]^{1/2},$$

où $S_n = \sum_{i=1}^n X_i$ et $V_n^2 = \sum_{i=1}^n X_i^2$. La loi asymptotique de cette statistique est donnée par celle de $\frac{S_n}{V_n}$. Ce sont ces dernières que nous qualifions de sommes autonormalisées. Des théorèmes de la limite centrale [Mal81], [GGM97] ont été démontrés pour ces sommes et des bornes de Berry-Esséen obtenues pour ces derniers [BG96], [Sha05], [BBG96], [BGT97]. Ces résultats de bornes d'erreurs stipulent que la distance de Kolmogorov entre la loi de $\frac{S_n}{V_n}$ et la gaussienne centrée réduite est inférieure à

$$C\left(B_n^{-2}\sum_{i=1}^N \mathbf{E}\left(X_i^2 1_{(|X_i|>B_n)}\right) + B_n^{-3}\sum_{i=1}^N \mathbf{E}\left(X_i^3 1_{(|X_i|\geq B_n)}\right)\right) \tag{9}$$

où $B_n = \sum_{i=1}^n \mathbf{E}(X_i^2)$ et où C est une constante universelle.

Le Chapitre 4 de cette thèse est constitué de la prépublication [BT11c] en collaboration avec Ciprian A. Tudor, soumis pour publication à *Séminaire de Probabilité*.

Dans cette partie de notre travail, nous mettons en oeuvre les techniques du calcul de Malliavin combinées à la méthode de Stein afin de determiner, dans un cadre gaussien, des bornes de Berry-Esséen pour le théorème de la limite centrale portant sur les sommes autonormalisées ([dlPLS09], page 53). Nous considérons comme variables aléatoires X_i les accroissements d'un processus de Wiener $(W_t; t \geq 0)$. Dans ce cas, la borne de Berry-Esséen (9) peut être réécrite de la manière suivante. Pour $2 < p \leq 3$,

$$\sup_{z\in\mathbb{R}}\left|\mathbf{P}(\frac{S_n}{V_n} \leq z) - \Phi(z)\right| \leq 25\mathbf{E}\left(|Z|^p\right) n^{1-\frac{p}{2}} \tag{10}$$

où Z est une variable aléatoire gaussienne centrée réduite et où Φ en est la fonction de répartition. En particulier, pour $p = 3$, on a

$$\sup_{z\in\mathbb{R}}\left|\mathbf{P}(\frac{S_n}{V_n} \leq z) - \Phi(z)\right| \leq 25\mathbf{E}\left(|Z|^3\right) n^{-\frac{1}{2}}. \tag{11}$$

Il est intérressant de noter que les bornes présentées ci-dessus ne sont valables que pour la distance de Kolmogorov. Nous étendons ce résultat à de nombreuses autres distances, telles que la distance de Wassertein, la distance de Fortet-Mourier ou encore la distance en variation totale. Nous ne pouvons bien sur pas nous attendre à trouver une meilleure vitesse de convergence que celle en $C\frac{1}{\sqrt{n}}$. En revanche, nous obtenons une expression explicite de la constante C mentionnée ci-dessus. Ces résultats sont obtenus par la combinaison de la méthode de Stein et du calcul de Malliavin. Nous déterminons l'expression exacte de la décomposition en chaos de Wiener de $\frac{S_n}{V_n}$, qui contrairement aux exemples contenus dans les travaux [BT11a], [NP09c], [NP09b], est une somme infinie d'intégrales stochastiques multiples. Cette décomposition chaotique nouvelle met également en lumière la nature uniforme par rapport aux chaos de la convergence de $\frac{S_n}{V_n}$ vers la gaussienne centrée réduite, dans le sens où chaque élément de la décomposition chaotique de $\frac{S_n}{V_n}$ converge lui même vers une gaussienne centrée réduite et que la vitesse de convergence est la même pour tous les chaos. Nous explicitons à présent les principaux résultats et commençons par le théorème de décomposition chaotique.

Théorème 10. *Soit* $F_n = \frac{S_n}{V_n}$ *et soit* $f : \mathbb{R}^n \to \mathbb{R}$ *définie par*

$$f(x_1, \cdots, x_n) = \frac{x_1 + \cdots + x_n}{(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}}.$$

Alors, pour tout $n \geq 2$, *on a*

$$F_n = \sum_{k\geq 0} \frac{1}{k!} \sum_{i_1,\cdots,i_k=1}^{n} a_{i_1,\cdots,i_k} I_k\left(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}\right)$$

avec

$$a_{i_1,\cdots,i_k} \stackrel{def}{=} \mathbf{E}\left(\frac{\partial^k f}{\partial x_{i_1}, \cdots, x_{i_k}} \left(W(\varphi_1), \cdots, W(\varphi_n)\right)\right).$$

Une partie importante de ce chapitre est ensuite consacrée à l'étude approfondie des coefficients $a_{i_1,\cdots,i_k}$ au sujet desquels nous obtenons tout d'abord le résultat suivant.

Lemme 1. *Si l'entier* k *est pair, alors*

$$a_{i_1,\cdots i_k} = 0.$$

Cette propriété étonnante permet d'ores et déjà d'eliminer une bonne partie des termes de la décomposition chaotique. Il reste ensuite à étudier plus avant les coefficients pour lesquels k est impair. Nous obtenons le résultat suivant.

Théorème 11. *Pour tout* $k \geq 0$ *et pour tout* $1 \leq i_1, \cdots, i_{2k+1} \leq n$, *soit* $d_r^\star, 1 \leq r \leq n$ *le nombre de fois que l'entier* r *apparait dans la suite* $\{i_1, \cdots, i_{2k+1}\}$. *Alors,*

$$a_{i_1,\cdots,i_{2k+1}} = \mathbf{E}\left[\frac{1}{V_n} W(\varphi_1)\mathbf{H}_{d_1^\star}\left(W(\varphi_1)\right)\mathbf{H}_{d_2^\star}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n^\star}\left(W(\varphi_n)\right)\right] \tag{12}$$

si il n'y a qu'un seul entier impair dans la suite $d_r^\star, 1 \leq r \leq n$. *Si il y en a plus qu'un, on a* $a_{i_1,\cdots,i_{2k+1}} = 0$.

Remarque 8. *Si on est dans le cas où $a_{i_1,\cdots,i_{2k+1}} \neq 0$, on peut réécrire $d_1^\star, d_2^\star, \cdots, d_n^\star$ comme $2d_1+1, 2d_2, \cdots, 2d_n$ et finalement, (12) se réécrit*

$$a_{i_1,\cdots,i_{2k+1}} = \mathbf{E}\left[\frac{1}{V_n} W(\varphi_1)\mathbf{H}_{2d_1+1}\left(W(\varphi_1)\right)\mathbf{H}_{2d_2}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{2d_n}\left(W(\varphi_n)\right)\right].$$

Nous avons également déterminé le comportement asymptotique des coefficients $a_{i_1,\cdots,i_{2k+1}}$, donnant lieu au résultat suivant.

Théorème 12. *Pour tout $1 \leq i_1, \cdots, i_{2k+1} \leq n$, le comportement asymptotique des coefficients $a_{i_1,\cdots,i_{2k+1}}$ est donné par, quand $n \to \infty$,*

$$\begin{aligned} a_{i_1,\cdots,i_{2k+1}} \quad &\sim \quad \frac{1}{k!}(2k-1)!!\frac{(2d_1+1)!(2d_2)!\cdots(2d_n)!}{(d_1!d_2!\cdots d_n!)^2} \\ &\times\ 2^{-2k}(-1)^k\left(\prod_{j=0}^{n}\sum_{l_j=0}^{d_j}(-1)^{l_j}C_{d_j}^{l_j}l_j^{d_j}\right)\frac{1}{n^{\frac{1}{2}+|A|}} \end{aligned} \tag{13}$$

où

$$A := \{2d_1+1, 2d_2, \cdots, 2d_n\} \setminus \{0,1\}$$

et où $|A|$ est le cardinal de l'ensemble A.

Ce résultat s'avère utile dans le calcul aussi explicite que possible de la constante de Berry-Esséen. Une fois la décomposition chaotique obtenue, nous passons au calcul de la borne de Berry-Esséen pour le théorème de la limite centrale pour les sommes autonormalisées. Pour cela, nous mettons en oeuvre la combinaison entre la méthode de Stein et le calcul de Malliavin introduite par Nourdin et Peccati dont le résultat principal utilisé ici porte sur la manière de borner la distance entre la loi $\mathcal{L}(F)$ d'une variable aléatoire F et la gaussienne centrée réduite par un terme faisant intervenir les opérateurs différentiels du calcul de Malliavin.

$$d(\mathcal{L}(F), \mathcal{N}(0,1)) \leq c\sqrt{\mathbf{E}\left(1 - \langle DF, D(-L)^{-1}F\rangle_{L^2([0,1])}\right)^2}.$$

Nous nous penchons donc sur le terme de borne $\mathbf{E}\left(1 - \langle DF, D(-L)^{-1}F\rangle_{L^2([0,1])}\right)^2$ et obtenons le résultat suivant.

Théorème 13. *Pour tout entier $n \geq 2$,*

$$\mathbf{E}\left(\left(\langle DF_n, D(-L)^{-1}F_n\rangle - 1\right)^2\right) \leq \frac{c_0}{n}$$

avec c_0 une constante combinatoire pouvant être écrite explicitement.

Le dernier résultat de ce chapitre porte sur le caractère uniforme de la convergence de notre suite autonormalisée vers la gaussienne centrée réduite. En effet, nous avons montré que chacun des chaos de la décomposition chaotique était convergent et convergeait vers une gaussienne centrée réduite à la même vitesse.

Corollaire 1. *Soit $J_m(F_n)$ la projection sur le $m^{\text{ième}}$ chaos de Wiener de la variable aléatoire F_n. Alors, pour tout $m \geq 1$, la suite $J_m(F_n)$ converge vers une gaussienne centrée réduite quand $n \to \infty$.*

0.4 Partie 3 : Régularité des solutions d'équations différentielles stochastiques rétrogrades

Dans la dernière partie de cette thèse, nous nous intéressons à des problématiques d'estimation de densité pour des solutions d'équations différentielles stochastiques et d'équations différentielles stochastiques rétrogrades. Nous détaillons ci après la notion d'équation différentielle stochastique rétrograde ainsi qu'un résultat nouveau de Nourdin et Viens [NV09] d'existence et de bornes de la densité d'une variable aléatoire de $\mathbb{D}^{1,2}$ (le domaine de l'opérateur de dérivation de Malliavin défini dans la Section 0.1.1).

0.4.1 Equations différentielles stochastiques rétrogrades

Une équation différentielle stochastique rétrograde (EDSR dans la suite) est une équation de la forme

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \tag{14}$$

avec la condition terminale (d'où le qualificatif de rétrograde) $Y_T = \xi$, où ξ est une variable aléatoire de carré intégrable (la solution d'une EDS par exemple). Résoudre une telle équation, c'est trouver un couple de processus $(Y_t, Z_t)_{t\in[0,T]}$ adaptés à la filtration du mouvement brownien $(W_t)_{0\leq t\leq T}$ et vérifiant (14). Les EDSR ont été introduites en 1973 par Bismut [Bis73] dans le cas où f est linéaire par rapport aux variables Y et Z. Il a fallu attendre le début des années 90 et le travail de Pardoux et Peng [PP90] pour avoir le premier résultat d'existence et d'unicité dans le cas où f n'est pas linéaire (voir aussi [PP92], [PP94] et [EKPQ97]). Depuis, de nombreux travaux ont été effectués ; la théorie n'a cessé de se développer en raison de ses relations étroites avec les mathématiques financières et les EDP (voir entre autre [GLW05], [GM10] ou encore [GL10]). Donnons un exemple emprunté à chacun de ces deux thèmes à titre illustratif. En finance, une question importante est de déterminer le prix d'une option. Prenons le cas le plus simple, à savoir celui du modèle de Black-Scholes et d'une option d'achat européenne. Le prix de ce produit financier, $(V_t)_{0\leq t\leq T}$ satisfait l'équation

$$dV_t = (rV_t + \theta Z_t)\, dt + Z_t dW_t,$$

où r est le taux d'inéret à court terme et θ la prime de risque de marché, avec la condition terminale $V_T = (S_T - K)^+$ où S_t est le prix de l'action sous-jacente et K une constante (le strike de l'option). Nous voyons que c'est une EDSR linéaire dans ce modèle simple mais qui peut être non-linéaire dans des modèles financiers plus compliqués. Venons en maintenant au deuxième exemple. Considérons l'EDP suivante

$$\partial_t u(t, x) + \frac{1}{2}\partial^2_{x,x} u(t, x) + f(u(t, x)) = 0, \quad u(T, x) = g(x).$$

Supposons que cette équation possède une solution régulière, u. Appliquons la formule d'Itô à $u(s, W_s)$; on obtient

$$\begin{aligned} du(s, W_s) &= \left\{\partial_s u(s, W_s) + \frac{1}{2}\partial^2_{x,x} u(s, W_s)\right\} ds + \partial_x u(s, W_s) dW_s \\ &= -f(u(s, W_s))ds + \partial_x u(s, W_s) dW_s. \end{aligned}$$

Nous obtenons encore une EDSR, qui est non linéaire si f ne l'est pas, en posant $Y_s = u(s, W_s)$ et $Z_s = \partial_x u(s, W_s)$ puisque

$$-dY_s = f(Y_s)ds - Z_s dW_s, \quad Y_T = g(W_T).$$

On remarquera que la condition terminale d'une EDSR peut être une fonction d'un processus de diffusion. La fonction f peut également dépendre, dans un cadre plus général, de ce processus de diffusion. C'est le cadre dans lequel se place cette partie de la thèse.

0.4.2 La formule de Nourdin-Viens

Dans [NV09], Corollaire 3.5, Nourdin et Viens ont donné des conditions suffisantes pour qu'une variable aléatoire différentiable au sens de Malliavin admette une densité et que celle-ci puisse être bornée inférieurement et supérieurement par des estimées gaussiennes.

Proposition 1. *Soit F une variable aléatoire de $\mathbb{D}^{1,2}$ et soit g une fonction définie pour tout $x \in \mathbb{R}$ par*

$$g(x) = \mathbf{E}\left(\langle DF, -DL^{-1}F\rangle_{L^2([0,T])}\Big|F - \mathbf{E}(F) = x\right). \tag{15}$$

Si il existe des constantes strictement positives $\gamma_{\min}, \gamma_{\max}$ telles que, pour tout $x \in \mathbb{R}$,

$$0 < \gamma_{\min}^2 \leqslant g(x) \leqslant \gamma_{\max}^2$$

presque sûrement, alors F possède une densité ρ qui satisfait, pour presque tout $z \in \mathbb{R}$

$$\frac{\mathbf{E}|F-\mathbf{E}(F)|}{2\,\gamma_{\max}^2}\exp\left(-\frac{(z-\mathbf{E}(F))^2}{2\gamma_{\min}^2}\right) \leqslant \rho(z) \leqslant \frac{\mathbf{E}|F-\mathbf{E}(F)|}{2\,\gamma_{\min}^2}\exp\left(-\frac{(z-\mathbf{E}(F))^2}{2\gamma_{\max}^2}\right).$$

Ce résultat peut être vu comme une alternative aux résultats de régularité ainsi que de minoration de densité contenus par exemple dans le Chapitre 2 de [Nua06]. De plus, Nourdin et Viens ont également donné un moyen pratique de calculer $g(x)$. Rappellons que $W = (W(\phi), \phi \in L^2([0,T]))$.

Proposition 2. *Soit F une variable aléatoire de $\mathbb{D}^{1,2}$ et on définit $DF = \Phi_F(W)$ pour une fonction mesurable $\Phi_F : \mathbb{R}^{L^2([0,T])} \to L^2([0,T])$. Alors, si $g(x)$ est définie par (15), on a*

$$g(x) = \int_0^\infty e^{-u}\,\mathbf{E}\left(\mathbf{E}'(\langle \Phi_F(W), \widetilde{\Phi_F^u}(W)\rangle_{L^2([0,T])})|F - \mathbf{E}(F) = x\right)du,$$

où $\widetilde{\Phi_F^u}(W) = \Phi_F(e^{-u}W + \sqrt{1-e^{-2u}}W')$, W' est une copie indépendante de W, et telle que W et W' soient définis sur l'espace produit $(\Omega\times\Omega', \mathcal{F}\otimes\mathcal{F}', \mathbb{P}\times\mathbb{P}')$ et $\mathbf{E}'$ est l'espérance mathématique par rapport à $\mathbb{P}'$.

0.4.3 Chapitre 5 : Estimées de densité pour les solutions d'EDS et d'EDSR unidimensionelles

Le résultat de Nourdin et Viens présenté dans le préambule de cette partie donne des conditions suffisantes pour prouver l'existence d'une densité pour une variable aléatoire de $\mathbb{D}^{1,2}$ ainsi que l'existence d'estimées (inférieures et supérieures) gaussiennes pour cette densité. Ce résultat a engendré plusieurs articles de recherche, comme ceux de Nualart et Quer-Sardanyons ([NQS09], [NQS11]), dans lesquels ils appliquent ce résultat aux solutions d'équations différentielles stochastiques aux dérivées partielles quasi-linéaires ainsi qu'à une classe d'équations stochastiques à bruit additif.

Le Chapitre 5 de cette thèse est constitué de la prépublication [AB11] en collaboration avec Omar Aboura, soumis pour publication à *Potential Analysis*.

Dans ce travail, nous utilisons l'approche de Nourdin et Viens pour prouver que, sous certaines conditions sur les coefficients, chaque composante de la solution (X_t, Y_t, Z_t) de l'équation différentielle stochastique rétrograde suivante

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \\ Y_t = \phi(X_T) + \int_t^T f(X_s, Y_s, Z_s)\, ds \int_t^T Z_s dW_s \end{cases}$$

possède une densité pour laquelle nous donnons également des estimées inférieures et supérieures. De plus, dans le cas de l'équation rétrograde à proprement dite, ces estimées sont gaussiennes. Cela implique de se pencher d'abord sur l'équation de diffusion régulant la condition terminale, puis sur l'EDSR dans un deuxième temps. Nous commençons donc par donner les résultats obtenus sur la partie diffusion de l'équation. Nous considérons pour cela l'équation différentielle stochastique unidimensionnelle suivante

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{16}$$

où $x_0 \in \mathbb{R}$, b et σ sont des fonctions assez régulières pour assurer l'existence et l'unicité de solutions et où $(W_t)_{t\geq 0}$ est un mouvement brownien standard sur $\mathbb{R}$. Afin d'énoncer notre premier résultat, nous nous plaçons dans le cadre d'hypothèses suivant. Nous considérons des fonctions b et σ C^2-lipschitz, ce qui assure l'existence et l'unicité de solutions à cette équation de diffusion. De plus, nous faisons les hypothèses suivantes :

$$\begin{cases} \textbf{H0 :} \begin{cases} \text{Pour tout } t > 0,\ \sigma(x) > 0 \text{ p.p sur le support de } X_t \\ \text{De plus, on suppose que } supp(X_t) \text{ est un intervalle indépendant de } t > 0 \end{cases} \\ \textbf{H1 :} \ \exists\ M_l \geq 0, |\,[b,\sigma]\,| \leq M_l|\sigma| \\ \textbf{H2 :} \ \exists\ M_{\sigma\sigma''} \geq 0, |\sigma\sigma''| \leq M_{\sigma\sigma''} \end{cases}$$

où $[b,\sigma]$ est le crochet de Lie de b et σ. On peut alors énoncer le résultat suivant portant sur l'existence d'une densité pour X_t ainsi que d'estimées inférieures et supérieures de cette densité.

Théorème 14. *Considerons l'équation (16) et soit G une primitive de $\frac{1}{\sigma}$. Sous les hypothèses ci-dessus, pour $t \in (0,T]$ la variable aléatoire X_t possède une densité ρ_{X_t}. De plus, il existe des constantes strictement positives c et C telles que, pour presque tout $x \in \mathbb{R}$, ρ_{X_t} vérifie :*

$$\rho_{X_t}(x) \geq \mathbf{1}_{supp(X_t)}(x)\frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)Ct}e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2ct}}$$

ainsi que

$$\rho_{X_t}(x) \leq \mathbf{1}_{supp(X_t)}(x)\frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)ct}e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2Ct}}.$$

Remarque 9. *Des résultats de ce type ont été démontrés par d'autres auteurs en utilisant des techniques différentes. On pourra notamment citer les travaux de Nualart et al. [CFN98], et de Bally et al. [BKH10], [Bal06].*

Fort de ce résultat, nous pouvons passer au cas des solutions (Y_t, Z_t) de l'EDSR. Il nous faut néanmoins renforcer quelques peu les hypothèses faites sur σ et b. Nous imposons maintenant

$$\begin{cases} \textbf{H3 :} \ \exists\ M_l \geq 0, \quad |[b,\sigma]| \leq M_l \\ \textbf{H4 :} \ \sigma \in B_0^{2,+}(\mathbb{R}) \end{cases}$$

où $B_0^{n,+}(\mathbb{R})$ est l'espace des fonctions $\mathcal{C}^n(\mathbb{R})$ bornées positivement telles que leurs dérivées jusqu'à l'ordre n soient bornées. D'un autre coté, les hypothèses que nous faisons sur la partie purement EDSR sont les suivantes.

$$\begin{cases} \textbf{H5 :} \ \exists\ c_{\phi'}, C_{\phi'}, \quad 0 < c_{\phi'} \leq |\phi'| \leq C_{\phi'} \\ \textbf{H6 :} \ \exists\ c_{f_x}, C_{f_x}, M_{f_y}, M_{f_z}, \quad \begin{Bmatrix} 0 < c_{f_x} \leq |f_x| \leq C_{f_x} \\ |f_y| \leq M_{f_y} \quad |f_z| \leq M_{f_z} \end{Bmatrix} \\ \textbf{H7 :} \ \forall u, v, \quad \phi'(u) f_x(v) > 0 \end{cases}$$

Sous ces hypothèses, nous pouvons nous intéresser à Y_t et completer les résultats obtenus par Antonelli et Kohatsu-Higa dans [AKH05]. Ces derniers ont montré, sous des hypothèses très similaires mais quelques peu plus souples, que Y_t possédait une densité. En revanche, les estimées contenues dans le théorème suivant sont non seulement nouvelles, mais expliquent aussi le besoin d'hypothèses quelques peu plus restrictives dans notre cas.

Théorème 15. *Sous les hypothèses ci-dessus, pour $t \in (0, T)$ la variable aléatoire Y_t possède une densité ρ_{Y_t}. De plus, il existe des constantes strictement positives c et C telles que, pour presque tout $y \in \mathbb{R}$, ρ_{Y_t} verifie :*

$$\frac{\mathbf{E}|Y_t - \mathbf{E}(Y_t)|}{2ct} \exp\left(-\frac{(y - \mathbf{E}(Y_t))^2}{2Ct}\right) \leq \rho_{Y_t}(y) \leq \frac{\mathbf{E}|Y_t - \mathbf{E}(Y_t)|}{2Ct} \exp\left(-\frac{(y - \mathbf{E}(Y_t))^2}{2ct}\right).$$

Nous présentons maintenant le dernier résultat de ce travail qui porte sur la dernière composante de la solution de l'EDSR : Z_t. Il n'existe pas, à notre connaissance, de résultat d'existence de densité ou à fortiori d'estimées de densité pour Z_t. C'est l'objet du dernier théorème. L'équation rétrograde que nous étudions pour le cas de Z_t est un peu moins générale que dans les autres cas. En effet, on impose que la dépendance de f en Z soit linéaire. On travaille donc avec

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \\ Y_t = \phi(X_T) + \int_t^T f^\star(X_s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s \end{cases}$$

où $f^\star(x, y, z) = f(x, y) + \alpha z$, $\alpha \in \mathbb{R}$. Nous précisons maintenant le cadre d'hypothèse dans lequel nous nous plaçons pour énoncer le dernier théorème. Sur la partie diffusion, nous demandons à ce que (**H1**) soit satisfaite ainsi que

$$\begin{cases} \textbf{H8 :} \ \sigma \in B_0^{3,+}(\mathbb{R}), \quad \sigma' \geq 0. \\ \textbf{H9 :} \ \exists\ M_l, M_{dl} \geq 0, \quad |[b,\sigma]| \leq M_l \sigma, \quad 0 \leq [\sigma, [\sigma, b]] \leq M_{dl}\sigma. \end{cases}$$

En ce qui concerne l'EDSR, nous faisons les hypothèses suivantes sur les fonctions ϕ et f, où $f : \mathbb{R}^2 \to \mathbb{R}$ ne depend pas de z :

$$\begin{cases} \textbf{H10 :} \text{ Il existe des constantes } c_{\phi'}, C_{\phi'}, C_{\phi''} \text{ telles que} \\ \qquad 0 < c_{\phi'} \leq \phi' \leq C_{\phi'},\ 0 < c_{\phi''} \leq \phi'' \leq C_{\phi''} \\ \textbf{H11 :} \text{ Il existe des constantes } m_{f_x}, M_{f_x}, M_{f_y}, M_{f_{xx}}, M_{f_{xy}}, M_{f_{yx}}, M_{f_{yy}} \text{ telles que} \\ \quad 0 < m_{f_x} \leq f_x \leq M_{f_x}, |f_y| \leq M_{f_y}, 0 \leq f_{xx} \leq M_{f_{xx}}, 0 \leq f_{xy} \leq M_{f_{xy}}, 0 \leq f_{yy} \leq M_{f_{yy}} \end{cases}$$

Remarquons que (**H10**) et (**H11**) impliquent (**H5**)-(**H7**). On a alors le résultat suivant.

Théorème 16. *Sous les hypothèses ci-dessus, pour* $t \in (0,T)$ *la variable aléatoire* Z_t *possède une densité* ρ_{Z_t}*. De plus, il existe des constantes strictement positives* c *et* C *telles que, pour presque tout* $z \in \mathbb{R}$, ρ_{Z_t} *vérifie :*

$$\frac{\mathbf{E}|Z_t - \mathbf{E}(Z_t)|}{2ct} \exp\left(-\frac{(z-\mathbf{E}(Z_t))^2}{2Ct}\right) \le \rho_{Z_t}(z) \le \frac{\mathbf{E}|Z_t - \mathbf{E}(Z_t)|}{2Ct} \exp\left(-\frac{(z-\mathbf{E}(Z_t))^2}{2ct}\right).$$

Première partie

Regression models and Malliavin calculus

Chapitre 1

Asymptotic theory for fractional regression models via Malliavin calculus

Solesne Bourguin * and Ciprian A. Tudor † ‡
Université Paris 1 and *Université Lille 1*

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Abstract

We study the asymptotic behavior as $n \to \infty$ of the sequence

$$S_n = \sum_{i=0}^{n-1} K(n^{\alpha} B_i^{H_1}) \left(B_{i+1}^{H_2} - B_i^{H_2} \right)$$

where B^{H_1} and B^{H_2} are two independent fractional Brownian motions, K is a kernel function and the bandwidth parameter α satisfies certain hypotheses in terms of H_1 and H_2. Its limiting distribution is a mixed normal law involving the local time of the fractional Brownian motion B^{H_1}. We use the techniques of the Malliavin calculus with respect to the fractional Brownian motion.

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Keywords : limit theorems, fractional Brownian motion, multiple stochastic integrals, Malliavin calculus, regression model, weak convergence.

1.1 Introduction

The motivation of our work comes from the econometric theory. Consider a regression model of the form

$$y_i = f(x_i) + u_i, \quad i \geq 0$$

∗. SAMM, Université de Paris 1 Panthéon-Sorbonne, 90, rue de Tolbiac, 75634, Paris, France. Email : solesne.bourguin@univ-paris1.fr

†. Laboratoire Paul Painlevé, Université de Lille 1, F-59655 Villeneuve d'Ascq, France. Email : tudor@math.univ-lille1.fr

‡. Associate member of the team Samm, Université de Paris 1 Panthéon-Sorbonne.

where $(u_i)_{i\geq 0}$ is the "error" and $(x_i)_{i\geq 0}$ is the regressor. The purpose is to estimate the function f based on the observation of the random variables y_i, $i \geq 0$. The conventional kernel estimate of $f(x)$ is

$$\hat{f}(x) = \frac{\sum_{i=0}^{n} K_h(x_i - x) y_i}{\sum_{i=0}^{n} K_h(x_i - x)}$$

where K is a nonnegative real kernel function satisfying $\int_{\mathbb{R}} K^2(y)dy = 1$ and $\int_{\mathbb{R}} yK(y)dy = 0$ and $K_h(s) = \frac{1}{h}K(\frac{s}{n})$. The bandwidth parameter $h \equiv h_n$ satisfies $h_n \to 0$ as $n \to \infty$. The asymptotic behavior of the estimator $\hat{f}$ is usually related to the behavior of the sequence

$$V_n = \sum_{i=1}^{n} K_h(x_i - x) u_i.$$

The limit in distribution as $n \to \infty$ of the sequence S_n has been widely studied in the literature in various situations. We refer, among others, to [KT01] and [KMT07] for the case where x_t is a recurrent Markov chain, to [WP09a] for the case where x_t is a partial sum of a general linear process, and [WP09b] for a more general situation. See also [PP01] or [Phi88]. An important assumption in the main part of the above references is the fact that u_i is a martingale difference sequence. In our work we will consider the following situation : we assume that the regressor $x_i = B_i^{H_1}$ is a fractional Brownian motion (fBm) with Hurst parameter $H_1 \in (0,1)$ and the error is $u_i = B_{i+1}^{H_2} - B_i^{H_2}$ where B^{H_2} is a fBm with $H_2 \in (0,1)$ and it is independent from B^{H_1}. In this case, our error process has no semimartingale property. We will also set $h_n = n^{-\alpha}$ with $\alpha > 0$. A supplementary assumption on α will be imposed later in terms of the Hurst parameters H_1 and H_2. The sequence V_n can be now written as

$$S_n(x) = \sum_{i=0}^{n} K(n^\alpha (B_i^{H_1} - x)) \left(B_{i+1}^{H_2} - B_i^{H_2} \right). \tag{1.1}$$

Our purpose is to give an approach based on stochastic calculus for this asymptotic theory. Recently, the stochastic integration with respect to the fractional Brownian motion has been widely studied. Various types of stochastic integrals, based on Malliavin calculus, Wick products or rough path theory have been introduced and change of variables formulas have been derived. We will use all these different techniques in our work. The general idea is as follows. Suppose that $x = 0$. We will first observe that the asymptotic behavior of the sequence S_n will be given by the sum

$$a_n = \sum_{i,j=0}^{n} K(n^\alpha B_i^{H_1}) K(n^\alpha B_j^{H_1}) \mathbf{E} \left((B_{i+1}^{H_2} - B_i^{H_2})(B_{j+1}^{H_2} - B_j^{H_2}) \right). \tag{1.2}$$

This is easy to understand since the conditional distribution of S_n given B^{H_1} is given by

$$(a_n)^{\frac{1}{2}} Z$$

where Z is a standard normal random variable. The double sum a_n can be decomposed into two parts : a "diagonal" part given by $\sum_{i=1}^{n} K^2(n^\alpha B_i^{H_1})$ and a "non-diagonal" part given by the terms with $i \neq j$. We will restrict ourselves to the situation where the diagonal part is dominant (in a sense that will be defined later) with respect to the non-diagonal part. This will imply a certain assumption on the bandwidth parameter α in terms of H_1 and H_2. We will therefore need to study the asymptotic behavior of

$$\langle S \rangle_n := \sum_{i=1}^{n} K^2(n^\alpha B_i^{H_1}). \tag{1.3}$$

(In the case $H_2 = \frac{1}{2}$ this is actually the bracket of S_n which is a martingale ; this motivates our choice of notation.) We will assume that the kernel K is the standard Gaussian kernel

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

This choice is motivated by the fact that $K^2(n^\alpha B_i^{H_1})$ can be decomposed into an orthogonal sum of multiple Wiener-Itô integrals (see [NV92b], [CNT01], [ELS+05]) and the Malliavin calculus can be used to treat the convergence of (1.3). Its limit in distribution will be after normalization the local time of the fractional Brownian motion denoted $cL^{H_1}(1,0)$, where c is positive constant. Consequently, we will find that the (renormalized) sequence S_n converges in law to a mixed normal random variable $cW_{L^{H_1}(1,0)}$ where W is a Brownian motion independent from B^{H_1} and c is a positive constant. The result is in concordance with the papers [WP09a], [WP09b].

But we also prove a stronger result : we show that the vector $(S_n, (G_t)_{t\geq 0})$ converges in the sense of finite dimensional distributions to the vector $(cW_{L^{H_1}(1,0)}, (G_t)_{t\geq 0})$, where c is a positive constant, for any stochastic process $(G_t)_{t\geq 0}$ independent from B^{H_1} and adapted to the filtration generated by B^{H_2} which satisfies some regularity properties in terms of the Malliavin calculus. We will say that S_n converges stably to its limit. To prove this stable convergence we will express S_n as a stochastic integral with respect to B^{H_2} and we will use the techniques of the Malliavin calculus. We will limit ourselves in this last section to the case $H_2 > \frac{1}{2}$.

We also mention that, although the error process B^{H_2} does not appear in the limit of (1.1), it governs the behavior of this sequence. Indeed, the parameter H_2 is involved in the renormalization of (1.1) and the stochastic calculus with respect to B^{H_2} is crucial in the proof of our main results.

We have organized our paper as follows : Section 2 contains the notations, definitions and results from the stochastic calculus that will be needed throughout our paper. In Section 3 we will find the renormalization order of the sequence (1.1), while Section 4 contains the result on the convergence of the "bracket" (1.3). In Section 5 we will prove the limit theorem in distribution for $S_n(0)$ and in Section 6 we will discuss the stable convergence of this sequence.

1.2 Preliminaries

Here we describe the elements from stochastic analysis that we will need in the paper. Consider $\mathcal{H}$ a real separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$, that is a centered Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$. Denote by I_n the multiple stochastic integral with respect to B (see [Nua06]). This I_n is actually an isometry between the Hilbert space $\mathcal{H}^{\odot n}$(symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}} \|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(B(\varphi))$ where $\varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1$ and H_n is the Hermite polynomial of degree $n \geq 1$

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as : for m, n positive integers,

$$\begin{aligned} \mathbf{E}\left(I_n(f) I_m(g)\right) &= n!\langle f, g\rangle_{\mathcal{H}^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}\left(I_n(f) I_m(g)\right) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{1.4}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \ldots, x_x) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by B can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n \geq 0} I_n(f_n) \tag{1.5}$$

where $f_n \in \mathcal{H}^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}\left[F\right]$.

Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n \geq 0} n I_n(f_n)$$

if F is given by (5.3).

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I - L)^{\frac{\alpha}{2}}\|_{L^p(\Omega)}$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in \mathcal{H}$)

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}\left(\mathcal{H}\right)$. The adjoint of D is denoted by δ and is called the divergence (or Skorohod) integral. It is a continuous operator from $\mathbb{D}^{\alpha,p}\left(\mathcal{H}\right)$ into $\mathbb{D}^{\alpha-1,p}$. We have the following duality relationship between D and δ

$$\mathbf{E}(F\delta(u)) = \mathbf{E}\langle DF, u\rangle_{\mathcal{H}} \text{ for every } F \text{ smooth.} \tag{1.6}$$

For adapted integrands, the divergence integral coincides with the classical Itô integral. We will use the notation

$$\delta(u) = \int_0^T u_s dB_s.$$

Let u be a stochastic process having the chaotic decomposition $u_s = \sum_{n \geq 0} I_n(f_n(\cdot, s))$ where $f_n(\cdot, s) \in \mathcal{H}^{\otimes n}$ for every s. One can prove that $u \in \operatorname{Dom} \delta$ if and only if $\tilde{f}_n \in \mathcal{H}^{\otimes(n+1)}$ for every $n \geq 0$, and $\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$ converges in $L^2(\Omega)$. In this case,

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \quad \text{and} \quad \mathbf{E}|\delta(u)|^2 = \sum_{n=0}^{\infty} (n+1)! \;\|\tilde{f}_n\|^2_{\mathcal{H}^{\otimes(n+1)}}.$$

In our work we will mainly consider divergence integrals with respect to a fractional Brownian motion. The fractional Brownian motion $(B_t^H)_{t\in[0,T]}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process starting from zero with covariance function

$$R^H(t,s) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad s,t \in [0,T].$$

In this case the space $\mathcal{H}_{\mathcal{H}}$ is the canonical Hilbert space of the fractional Brownian motion which is defined as the closure of the linear space generated by the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}_H} = R^H(t,s), \qquad s,t \in [0,T].$$

1.3 Renormalization of the sequence S_n

As we mentioned in the introduction, we will assume throughout the paper that $x = 0$ in (1.1), then

$$S_n := S_n(0) = \sum_{i=0}^{n-1} K(n^\alpha B_i^{H_1})(B_{i+1}^{H_2} - B_i^{H_2}). \tag{1.7}$$

We compute in this part the L^2-norm of S_n in order to renormalize it. We have

$$\begin{aligned}
\mathbf{E}\left(S_n^2\right) &= \mathbf{E}\left(\sum_{i,j=0}^{n-1} K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})(B_{i+1}^{H_2} - B_i^{H_2})(B_{j+1}^{H_2} - B_j^{H_2})\right)\\
&= \mathbf{E}\left(\sum_{i=0}^{n-1} K^2(n^\alpha B_i^{H_1})(B_{i+1}^{H_2} - B_i^{H_2})^2\right)\\
&\quad +\mathbf{E}\left(\sum_{i\neq j}^{n-1} K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})(B_{i+1}^{H_2} - B_i^{H_2})(B_{j+1}^{H_2} - B_j^{H_2})\right)\\
&= T' + T''.
\end{aligned}$$

The summand T' will be called the "diagonal" term while the summand T'' will be called "the non-diagonal" term. We will analyze each of them separately. Concerning T' we have

Lemma 1. *As $n \to +\infty$,*

$$n^{\alpha+H_1-1}T' \underset{n\to+\infty}{\longrightarrow} C_1 = \frac{1}{2\pi\sqrt{2}(1-H_1)}. \tag{1.8}$$

Proof : Through the independence of $\left(B_t^{H_1}\right)_{t\geq 0}$ and $\left(B_t^{H_2}\right)_{t\geq 0}$,

$$T' = \sum_{i=0}^{n-1} \mathbf{E}\left(K^2(n^\alpha B_i^{H_1})\right)\mathbf{E}\left((B_{i+1}^{H_2} - B_i^{H_2})^2\right).$$

Since $\mathbf{E}\left((B_{i+1}^{H_2} - B_i^{H_2})^2\right) = 1$,

$$T' = \sum_{i=0}^{n-1} \mathbf{E}\left(K^2(n^\alpha B_i^{H_1})\right) = \sum_{i=0}^{n-1} \mathbf{E}\left(\frac{1}{2\pi} e^{-n^{2\alpha}i^{2H_1}Z^2}\right)$$

where Z is a standard normal random variable. Recall that, if Z is a standard normal random variable, and if $1+2c>0$

$$\mathbf{E}\left(e^{-cZ^2}\right)=\frac{1}{\sqrt{1+2c}} \tag{1.9}$$

consequently,

$$T'=\sum_{i=0}^{n-1}\frac{1}{2\pi\sqrt{1+2n^{2\alpha}i^{2H_1}}}.$$

As $n\to+\infty$, T' behaves as such

$$\begin{aligned}\sum_{i=0}^{n-1}\frac{1}{2\pi\sqrt{1+2n^{2\alpha}i^{2H_1}}} &\sim \frac{n^{-\alpha}}{2\pi\sqrt{2}}\sum_{i=0}^{n-1}i^{-H_1}\sim\frac{n^{-\alpha-H_1+1}}{2\pi\sqrt{2}}\frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^{-H_1}\\ &\sim \frac{n^{-\alpha-H_1+1}}{2\pi\sqrt{2}}\int_0^1 x^{-H_1}dx=\frac{n^{-\alpha-H_1+1}}{2\pi\sqrt{2}(1-H_1)}.\end{aligned}$$

The sign "$\sim$" means that the left-hand side and the right-hand side have the same limit as $n\to+\infty$. We will use this notation throughout the paper. ■

We will now compute the term T''. To do so, we will need the following Lemma (lemma 3.1 p. 122 in [YLY09]).

Lemma 2. *For every $s,r\in[0,T]$, $s\geq r$ and $0<H<1$ we have*

$$s^{2H}r^{2H}-\mu^2\geq\tau(s-r)^{2H}r^{2H} \tag{1.10}$$

where $\mu=\mathbf{E}(B_s^HB_r^H)$ and $\tau>0$ is a constant.

Concerning the non-diagonal term of $\mathbf{E}\left(S_n^2\right)$ the following holds

Lemma 3. *Suppose that*

$$\alpha-4H_2+H_1+2>0. \tag{1.11}$$

Then, as $n\to+\infty$,

$$n^{\alpha+H_1-1}T''\underset{n\to+\infty}{\longrightarrow}0. \tag{1.12}$$

Proof : Using again the independence of $\left(B_t^{H_1}\right)_{t\geq0}$ and $\left(B_t^{H_2}\right)_{t\geq0}$

$$\begin{aligned}T'' &= \sum_{i\neq j}^{n-1}\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right)\mathbf{E}\left((B_{i+1}^{H_2}-B_i^{H_2})(B_{j+1}^{H_2}-B_j^{H_2})\right)\\ &= \frac{1}{2}\sum_{i\neq j}^{n-1}\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right)f_{H_2}(i,j)\end{aligned}$$

where

$$f_{H_2}(i,j)=\frac{1}{2}\left[|i-j+1|^{2H_2}+|i-j-1|^{2H_2}-2\,|i-j|^{2H_2}\right]. \tag{1.13}$$

We need to evaluate the expectation $\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right)$. Let $\Gamma = \begin{pmatrix} i^{2H_1} & R(i,j) \\ R(i,j) & j^{2H_1}\end{pmatrix}$ be the covariance matrix of $\left(B_i^{H_1}, B_j^{H_1}\right)$. We have $|\Gamma| = (ij)^{2H_1} - R^2(i,j)$ and $\Gamma^{-1} = \frac{1}{|\Gamma|}\begin{pmatrix} j^{2H_1} & -R(i,j) \\ -R(i,j) & i^{2H_1}\end{pmatrix}$. The density of $\left(B_i^{H_1}, B_j^{H_1}\right)$ is then

$$f(x,y) = \frac{1}{2\pi\sqrt{|\Gamma|}} e^{-\frac{1}{2|\Gamma|}(j^{2H_1}x^2 - 2R(i,j)xy + i^{2H_1}y^2)}. \tag{1.14}$$

We obtain

$$\begin{aligned}
&\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right)\\
=\;& \frac{1}{(2\pi)^2\sqrt{|\Gamma|}}\int_{\mathbb{R}^2} e^{-\frac{n^{2\alpha}x^2}{2}} e^{-\frac{n^{2\alpha}y^2}{2}} e^{-\frac{1}{2|\Gamma|}(j^{2H_1}x^2 - 2R(i,j)xy + i^{2H_1}y^2)} dxdy\\
=\;& \frac{1}{(2\pi)^2\sqrt{|\Gamma|}}\int_{\mathbb{R}} e^{-\frac{n^{2\alpha}y^2}{2}} e^{-\frac{i^{2H_1}y^2}{2|\Gamma|}} \int_{\mathbb{R}} e^{-\frac{n^{2\alpha}x^2}{2}} e^{-\frac{1}{2|\Gamma|}(j^{2H_1}x^2 - 2R(i,j)xy)} dxdy\\
=\;& \frac{1}{(2\pi)^2\sqrt{|\Gamma|}}\int_{\mathbb{R}} e^{-\frac{y^2}{2}\left[n^{2\alpha}+\frac{i^{2H_1}}{|\Gamma|}\right]} \int_{\mathbb{R}} e^{-\frac{1}{2}\left[x^2\left(n^{2\alpha}+\frac{j^{2H_1}}{|\Gamma|}\right)-\frac{2R(i,j)}{|\Gamma|}xy\right]} dxdy\\
=\;& \frac{1}{(2\pi)^2\sqrt{|\Gamma|}}\int_{\mathbb{R}} e^{-\frac{y^2}{2}\left[n^{2\alpha}+\frac{i^{2H_1}}{|\Gamma|}\right]} \int_{\mathbb{R}} e^{-\frac{\left(n^{2\alpha}+\frac{j^{2H_1}}{|\Gamma|}\right)}{2}\left[x^2-\frac{2R(i,j)}{n^{2\alpha}|\Gamma|+j^{2H_1}}xy\right]} dxdy\\
=\;& \frac{1}{(2\pi)^2\sqrt{|\Gamma|}}\int_{\mathbb{R}} e^{-\frac{y^2}{2}\left[n^{2\alpha}+\frac{i^{2H_1}}{|\Gamma|}\right]}\\
\times\;& \int_{\mathbb{R}} e^{-\frac{\left(n^{2\alpha}+\frac{j^{2H_1}}{|\Gamma|}\right)}{2}\left[\left(x-\frac{R(i,j)}{n^{2\alpha}|\Gamma|+j^{2H_1}}y\right)^2-\frac{R^2(i,j)}{(n^{2\alpha}|\Gamma|+j^{2H_1})^2}y^2\right]} dxdy\\
=\;& \frac{1}{(2\pi)^2\sqrt{|\Gamma|}}\int_{\mathbb{R}} e^{-\frac{y^2}{2}\left[n^{2\alpha}+\frac{i^{2H_1}}{|\Gamma|}\right]} e^{-\frac{\left(n^{2\alpha}+\frac{j^{2H_1}}{|\Gamma|}\right)}{2}\frac{R^2(i,j)y^2}{\left(n^{2\alpha}+\frac{j^{2H_1}}{|\Gamma|}\right)^2|\Gamma|^2}}\\
\times\;& \int_{\mathbb{R}} e^{-\frac{\left(n^{2\alpha}+\frac{j^{2H_1}}{|\Gamma|}\right)}{2}\left(x-\frac{R(i,j)}{n^{2\alpha}|\Gamma|+j^{2H_1}}y\right)^2} dxdy\\
=\;& \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{|\Gamma|}}\frac{\sqrt{|\Gamma|}}{\sqrt{n^{2\alpha}|\Gamma|+j^{2H_1}}}\int_{\mathbb{R}} e^{-\frac{1}{2}y^2\left[\frac{(n^{2\alpha}|\Gamma|+i^{2H_1})(n^{2\alpha}|\Gamma|+j^{2H_1})-R^2(i,j)}{|\Gamma|(n^{2\alpha}|\Gamma|+j^{2H_1})}\right]} dy.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right)\\
=\;& \frac{1}{2\pi\sqrt{n^{2\alpha}|\Gamma|+j^{2H_1}}}\frac{\sqrt{|\Gamma|}\sqrt{(n^{2\alpha}|\Gamma|+j^{2H_1})}}{\sqrt{(n^{2\alpha}|\Gamma|+i^{2H_1})(n^{2\alpha}|\Gamma|+j^{2H_1})-R^2(i,j)}}\\
=\;& \frac{\sqrt{|\Gamma|}}{2\pi\sqrt{(n^{2\alpha}|\Gamma|+i^{2H_1})(n^{2\alpha}|\Gamma|+j^{2H_1})-R^2(i,j)}}\\
=\;& \frac{1}{2\pi\sqrt{n^{4\alpha}|\Gamma|+n^{2\alpha}j^{2H_1}+n^{2\alpha}i^{2H_1}+1}}.
\end{aligned}$$

Suppose that $i > j$. We use Lemma 2 to bound $|\Gamma| = i^{2H_1} j^{2H_1} - R^2(i,j)$ from below. Therefore

$$\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right) \leq \frac{1}{2\pi\sqrt{n^{4\alpha}\tau(i-j)^{2H_1}j^{2H_1} + n^{2\alpha}(i^{2H_1}+j^{2H_1})}}.$$

Since $a^2+b^2 \geq 2ab$ with $a^2 = n^{4\alpha}\tau(i-j)^{2H_1}j^{2H_1}$ and $b^2 = n^{2\alpha}(i^{2H_1}+j^{2H_1})$

$$\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right) \leq \frac{1}{2\pi\sqrt{2\sqrt{\tau}n^{2\alpha}(i-j)^{H_1}j^{H_1}\sqrt{n^{2\alpha}(i^{2H_1}+j^{2H_1})}}}$$

and using the same inequality as above for $a^2 = i^{2H_1}$ and $b^2 = j^{2H_1}$

$$\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right) \leq \frac{n^{-\frac{3\alpha}{2}}}{2\pi\sqrt{2}\tau^{\frac{1}{4}}(i-j)^{\frac{H_1}{2}}j^{\frac{3H_1}{4}}i^{\frac{H_1}{4}}}. \tag{1.15}$$

Since $f_{H_2}(i,j)$ behaves as $H_2(2H_2-1)|i-j|^{2H_2-2}$ when $i-j\to\infty$, we can assert that

$$T'' \sim \frac{H_2(2H_2-1)}{2}\sum_{i\neq j}^{n-1}\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right)|i-j|^{2H_2-2}.$$

Using (1.15), we can write

$$\sum_{i\neq j}^{n-1}\mathbf{E}\left(K(n^\alpha B_i^{H_1})K(n^\alpha B_j^{H_1})\right)|i-j|^{2H_2-2} \lesssim \sum_{i>j}^{n-1}\frac{n^{-\frac{3\alpha}{2}}}{2\pi\sqrt{2}\tau^{\frac{1}{4}}(i-j)^{\frac{H_1}{2}}j^{\frac{3H_1}{4}}i^{\frac{H_1}{4}}}|i-j|^{2H_2-2}$$

and consequently

$$\begin{aligned} T'' &\lesssim \frac{H_2(2H_2-1)}{4\pi\sqrt{2}\tau^{\frac{1}{4}}}n^{-\frac{3\alpha}{2}}n^{2H_2-\frac{H_1}{2}-2}n^{-\frac{3H_1}{4}}n^{-\frac{H_1}{4}}n^2\underbrace{\frac{1}{n^2}\sum_{i>j}^{n-1}\frac{\left(\frac{i-j}{n}\right)^{2H_2-\frac{H_1}{2}-2}}{\left(\frac{j}{n}\right)^{\frac{3H_1}{4}}}\left(\frac{j}{n}\right)^{\frac{H_1}{4}}}_{\underset{n\to+\infty}{\longrightarrow}C(H_1,H_2)>0} \\ &\lesssim \frac{H_2(2H_2-1)C(H_1,H_2)}{4\pi\sqrt{2}\tau^{\frac{1}{4}}}n^{-\frac{3\alpha}{2}+2H_2-\frac{3H_1}{2}}. \end{aligned} \tag{1.16}$$

It follows that under condition (1.11) $n^{\alpha+H_1-1}T''$ converges to zero as $n\to\infty$. ∎

As a consequence of Lemmas 6 and 3 we obtain the following L^2- norm estimate for S_n.

Proposition 3. *Suppose that condition (1.11) holds. Then, as $n\to\infty$*

$$n^{\alpha+H_1-1}\mathbf{E}\left(S_n^2\right) \to C_1 = \frac{1}{2\pi\sqrt{2}(1-H_1)}.$$

The condition (1.11) will be discussed more thoroughly later (Remark 1, Section 5).

1.4 The limit in distribution of $\langle S\rangle_n$

Proposition 3 implies that the diagonal part of S_n^2 is dominant in relation to the non-diagonal part, in the sense that this diagonal part is responsable for the renormalization order of S_n^2 which is $n^{\alpha+H_1-1}$. As a consequence we need to study the limit distribution of $n^{\alpha+H_1-1}\langle S\rangle_n = n^{\alpha+H_1-1}\sum_{i=0}^{n-1} K^2(n^\alpha B_i^{H_1})$. Using the self-similarity property of the fractional Brownian motion we have

$$n^{\alpha+H_1-1}\sum_{i=0}^{n-1} K^2(n^\alpha B_i^{H_1}) = n^{\alpha+H_1-1}\sum_{i=0}^{n-1} K^2(n^{\alpha+H_1} B_{\frac{i}{n}}^{H_1}).$$

The limit of the above sequence is linked to the local time of the fractional Brownian motion B^{H_1}. For any $t \geq 0$ and $x \in \mathbb{R}$ we define $L^{H_1}(t,x)$ as the density of the occupation measure (see [Ber74], [GH80])

$$\mu_t(A) = \int_0^t 1_A(B_s^{H_1})ds, \qquad A \in \mathcal{B}(\mathbb{R}).$$

The local time $L^{H_1}(t,x)$ satisfies the occupation time formula

$$\int_0^t f(B_s^{H_1})ds = \int_{\mathbb{R}} L^{H_1}(t,x)f(x)dx \tag{1.17}$$

for any measurable function f. The local time is Hölder continuous with respect to t and with respect to x (for the sake of completeness $L^{H_1}(t,x)$ has Hölder continuous paths of order $\delta < 1-H$ in time and of order $\gamma < \frac{1-H}{2H}$ in the space variable (see Table 2 in [GH80])). Moreover, it admits a bicontinuous version with respect to (t,x).

Below, we give an important convergence result that will be necessary in proving the main result of this section.

Proposition 4. *The following convergence in distribution result holds*

$$n^{\alpha+H_1}\left(\frac{1}{n}\sum_{i=0}^{n-1} K^2(n^{\alpha+H_1}B_{\frac{i}{n}}^{H_1}) - \int_0^1 K^2(n^{\alpha+H_1}B_s^{H_1})ds\right) \underset{n\to+\infty}{\longrightarrow} 0. \tag{1.18}$$

Proof : Fix $\varepsilon > 0$. Let $p_\varepsilon(x)$ be the Gaussian kernel with variance $\varepsilon > 0$ defined by $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{x^2}{2\varepsilon}}$. Note that for every $s \geq 0$

$$\sqrt{\pi}n^{\alpha+H_1}K^2(n^{\alpha+H_1}B_s^{H_1}) = \frac{1}{2}p_{\frac{1}{2n^{2(\alpha+H_1)}}}(B_s^{H_1}). \tag{1.19}$$

Using (1.19), we can write the left-hand side of (1.18) as

$$\begin{aligned}
&\sqrt{\pi}n^{\alpha+H_1}\left(\int_0^1 K^2(n^{\alpha+H_1}B_s^{H_1})ds - \frac{1}{n}\sum_{i=0}^{n-1}K^2(n^{\alpha+H_1}B_{\frac{i}{n}}^{H_1})\right)\\
=\ & \frac{1}{2}\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\left(p_{\frac{1}{2}n^{-2(\alpha+H_1)}}(B_s^{H_1}) - p_{\frac{1}{2}n^{-2(\alpha+H_1)}}(B_{\frac{i}{n}}^{H_1})\right)ds\\
=\ & \frac{1}{2}\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\left(p_{\frac{1}{2}n^{-2(\alpha+H_1)}}(B_s^{H_1}) - p_\varepsilon(B_s^{H_1})\right)ds\\
&+\frac{1}{2}\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\left(p_\varepsilon(B_s^{H_1}) - p_\varepsilon(B_{\frac{i}{n}}^{H_1})\right)ds\\
&+\frac{1}{2}\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\left(p_\varepsilon(B_{\frac{i}{n}}^{H_1}) - p_{\frac{1}{2}n^{-2(\alpha+H_1)}}(B_{\frac{i}{n}}^{H_1})\right)ds := \frac{1}{2}(a_n^{(1)} + a_n^{(2)} + a_n^{(3)}).
\end{aligned}$$

We will now estimate the three terms above and we will show that each of them converges to zero (in some sense). Let us first handle the term $a_n^{(1)}$. We have

$$a_n^{(1)} = \int_0^1 p_{\frac{1}{2}n^{-2(\alpha+H_1)}}(B_s^{H_1})ds - \int_0^1 p_\varepsilon(B_s^{H_1})ds.$$

It follows from [NV92b] or [ELS+05] that

$$\int_0^1 p_\varepsilon(B_s^{H_1})ds \to_{\varepsilon\to 0} \int_0^1 \delta_0(B_s^{H_1})ds = L^{H_1}(1,0) \tag{1.20}$$

in $L^2(\Omega)$ and almost surely, where $L^{H_1}(1,0)$ is the local time of the fractional Brownian motion. Therefore $a_n^{(1)}$ clearly converges to zero as $\varepsilon \to 0$ and $n \to \infty$. The term $a_n^{(2)}$ can be expressed as

$$a_n^{(2)} = -\left(\frac{1}{n}\sum_{i=0}^{n-1} p_\varepsilon(B_{\frac{i}{n}}^{H_1}) - \int_0^1 p_\varepsilon(B_s^{H_1})ds\right) \tag{1.21}$$

and for every $\varepsilon > 0$ it converges almost surely to zero as $n \to \infty$ using the Riemann sum convergence. Let us now handle the term $a_n^{(3)}$ given by

$$a_n^{(3)} = \frac{1}{n}\sum_{i=0}^{n-1}\left(p_\varepsilon(B_{\frac{i}{n}}^{H_1}) - p_{\frac{1}{2}n^{-2(\alpha+H_1)}}(B_{\frac{i}{n}}^{H_1})\right). \tag{1.22}$$

We will treat this term by using the chaos decomposition of the Gaussian kernel applied to random variables in the first Wiener chaos. Recall that (see [CNT01], [HØ02], [IW94], [NV92a]) for every $\varphi \in \mathcal{H}_{H_1}$ ($\mathcal{H}_{H_1}$ is the canonical Hilbert space associated with the Gaussian process B^{H_1}),

$$p_\varepsilon(B^{H_1}(\varphi)) = \sum_{m\geq 0} C_m I_{2m}\left(\varphi^{\otimes 2m}\right)\frac{1}{\left(\|\varphi\|^2_{\mathcal{H}_1} + \varepsilon\right)^{m+\frac{1}{2}}} \tag{1.23}$$

where $C_m = \frac{(-1)^m}{\sqrt{2\pi}2^m m!}$.

Using this chaos decomposition, we can write $p_\varepsilon(B_{\frac{i}{n}}^{H_1}) - p_{\frac{1}{2}n^{-2(\alpha+H_1)}}(B_{\frac{i}{n}}^{H_1})$ as

$$\begin{aligned}
&p_\varepsilon(B_{\frac{i}{n}}^{H_1}) - p_{\frac{1}{2}n^{-2(\alpha+H_1)}}(B_{\frac{i}{n}}^{H_1}) \\
= &\sum_{m\geq 0} C_m I_{2m}\left(1_{[0,\frac{i}{n}]}^{\otimes 2m}\right)\left(\frac{1}{\left(\left(\frac{i}{n}\right)^{2H_1}+\varepsilon\right)^{m+\frac{1}{2}}} - \frac{1}{\left(\left(\frac{i}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)^{m+\frac{1}{2}}}\right) \\
= &\sum_{m\geq 0} C_m I_{2m}\left(1_{[0,\frac{i}{n}]}^{\otimes 2m}\right)\left(\frac{i}{n}\right)^{-2H_1\left(m+\frac{1}{2}\right)} d_{i,\varepsilon,n,m}
\end{aligned}$$

where

$$d_{i,\varepsilon,n,m} = \left(\left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\varepsilon\right)}\right)^{m+\frac{1}{2}} - \left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)}\right)^{m+\frac{1}{2}}\right).$$

We will show that $a_n^{(3)}$ converges to zero in $L^2(\Omega)$ as $n\to\infty$ and $\varepsilon\to 0$. From (1.22) one can easily see that the diagonal part of $a_n^{(3)}$ converges to zero. We can also see, from the expression of $a_n^{(3)}$, that the summands with $j=0$ vanish. Then, by using the orthogonality of multiple stochastic integrals([Nua06]), we obtain

$$\begin{aligned}\mathbf{E}(a_n^{(3)})^2 &\sim \frac{1}{n^2}\sum_{m\geq 0} C_m^2(2m)!\sum_{i,j\geq 1;i\neq j}^{n-1}\langle 1_{[0,\frac{i}{n}]},1_{[0,\frac{j}{n}]}\rangle_{\mathcal{H}_1}^{2m}\left(\frac{i}{n}\right)^{-2H_1(m+\frac{1}{2})}\left(\frac{j}{n}\right)^{-2H_1(m+\frac{1}{2})}\\ &\quad \times d_{i,\varepsilon,n,m}d_{j,\varepsilon,n,m}.\end{aligned}$$

We can also write

$$\begin{aligned}\mathbf{E}(a_n^{(3)})^2 &\sim \frac{1}{n^2}\sum_{m\geq 0} C_m^2(2m)!\sum_{i,j\geq 1,i\neq j}^{n-1}R_{H_1}\left(\frac{i}{n},\frac{j}{n}\right)^{2m}\left(\frac{i}{n}\right)^{-2H_1(m+\frac{1}{2})}\left(\frac{j}{n}\right)^{-2H_1(m+\frac{1}{2})}\\ &\quad \times d_{i,\varepsilon,n,m}d_{j,\varepsilon,n,m}\\ &:= \sum_{m\geq 0} C_m^2(2m)!A_m(\varepsilon,n).\end{aligned}$$

where

$$A_m(\varepsilon,n)=\frac{1}{n^2}\sum_{i,j\geq 1;i\neq j}^{n-1}R_{H_1}\left(\frac{i}{n},\frac{j}{n}\right)^{2m}\left(\frac{i}{n}\right)^{-2H_1(m+\frac{1}{2})}\left(\frac{j}{n}\right)^{-2H_1(m+\frac{1}{2})}d_{i,\varepsilon,n,m}d_{j,\varepsilon,n,m}.$$

We can now claim that, for every fixed $m\geq 1$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}A_m(\varepsilon,n)=0. \tag{1.24}$$

Indeed, for every $m\geq 0$, we get

$$\begin{aligned}|d_{i,\varepsilon,n,m}| &= \left|\left(\left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\varepsilon\right)}\right)^{m+\frac{1}{2}}-1+1-\left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)}\right)^{m+\frac{1}{2}}\right)\right|\\ &\leq \left|1-\left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\varepsilon\right)}\right)^{m+\frac{1}{2}}\right|+\left|1-\left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)}\right)^{m+\frac{1}{2}}\right|\\ &\leq \left|1-\left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\varepsilon\right)}\right)^{m+1}\right|+\left|1-\left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)}\right)^{m+1}\right|\\ &= c_m\left(\left|\left(\frac{\varepsilon}{\left(\left(\frac{i}{n}\right)^{2H_1}+\varepsilon\right)}\right)\right|+\left|\left(\frac{n^{-2(\alpha+H_1)}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)}\right)\right|\right).\end{aligned}$$

Now, for every i, n, m, we have $\lim_{\varepsilon\to 0}\left|\left(\frac{\varepsilon}{\left(\left(\frac{i}{n}\right)^{2H_1}+\varepsilon\right)}\right)\right| = 0$ and for every $i \geq 1$,

$$\begin{aligned}
\left|\left(\frac{n^{-2(\alpha+H_1)}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)}\right)\right| &\leq \left|\left(\frac{n^{-2(\alpha+H_1)}}{\left(\left(\frac{1}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)}\right)\right| \\
&\leq c\frac{n^{2H_1}}{n^{2(\alpha+2H_1)}+n^{2H_1}} \underset{n\to+\infty}{\longrightarrow} 0
\end{aligned}$$

because $\alpha > 0$.

Furthermore, we know that

$$\frac{1}{n^2}\sum_{i,j=0}^{n-1} R_{H_1}\left(\frac{i}{n},\frac{j}{n}\right)^{2m}\left(\frac{i}{n}\right)^{-2H_1(m+\frac{1}{2})}\left(\frac{j}{n}\right)^{-2H_1(m+\frac{1}{2})}$$

converges as $n \to \infty$ to $\int_0^1\int_0^1 R(u,v)^{2m}(uv)^{-2H_1(m+\frac{1}{2})}dudv$. Since this quantity is finite ([CNT01] and [ELS+05]), it implies (1.24).

We will now prove that

$$\sum_{m\geq 0} C_m^2(2m)!\sup_{n,\varepsilon}|A_m(\varepsilon,n)| < \infty. \tag{1.25}$$

Relation (1.24) and (1.25) will imply the convergence of $a_n^{(3)}$ to zero in $L^2(\Omega)$. We need to find an upper bound for the terms $|d_{i,\varepsilon,n,m}|$ and $|d_{j,\varepsilon,n,m}|$ in order to continue.

$$\begin{aligned}
d_{i,\varepsilon,n,m} &= \left(\left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\varepsilon\right)}\right)^{m+\frac{1}{2}} - \left(\frac{\left(\frac{i}{n}\right)^{2H_1}}{\left(\left(\frac{i}{n}\right)^{2H_1}+\frac{1}{2}n^{-2(\alpha+H_1)}\right)}\right)^{m+\frac{1}{2}}\right) \\
&= \left(\left(\frac{1}{(1+\varepsilon n^{2H}i^{-2H})}\right)^{m+\frac{1}{2}} - \left(\frac{1}{\left(1+\frac{1}{2}n^{-2\alpha}i^{-2H}\right)}\right)^{m+\frac{1}{2}}\right).
\end{aligned}$$

One can note that

$$0 \leq \left(\frac{1}{(1+\varepsilon n^{2H}i^{-2H})}\right)^{m+\frac{1}{2}} \leq 1 \quad \text{and} \quad 0 \leq \left(\frac{1}{\left(1+\frac{1}{2}n^{-2\alpha}i^{-2H}\right)}\right)^{m+\frac{1}{2}} \leq 1$$

because $\varepsilon n^{2H}i^{-2H} > 0$. From the above inequalities, we can deduce that

$$-1 \leq \left(\frac{1}{(1+\varepsilon n^{2H}i^{-2H})}\right)^{m+\frac{1}{2}} - \left(\frac{1}{\left(1+\frac{1}{2}n^{-2\alpha}i^{-2H}\right)}\right)^{m+\frac{1}{2}} \leq 1$$

and finally,

$$|d_{i,\varepsilon,n,m}| \leq 1 \quad \text{and} \quad |d_{j,\varepsilon,n,m}| \leq 1.$$

By bounding from above the terms $|d_{i,\varepsilon,n,m}|$ and $|d_{j,\varepsilon,n,m}|$ by 1 in $\sum_{m\geq 0} C_m^2(2m)!\sup_{n,\varepsilon}|A_m(\varepsilon,n)|$ we obtain that

$$\begin{aligned}
&\sum_{m\geq 0} C_m^2(2m)!\sup_{n,\varepsilon}|A_m(\varepsilon,n)|\\
&\leq \sum_{m\geq 0} C_m^2(2m)!\sup_n \frac{1}{n^2}\sum_{i,j\geq 1, i\neq j}^{n-1} R_{H_1}\left(\frac{i}{n},\frac{j}{n}\right)^{2m}\left(\frac{i}{n}\right)^{-2H_1(m+\frac{1}{2})}\left(\frac{j}{n}\right)^{-2H_1(m+\frac{1}{2})}\\
&= \sum_{m\geq 0} C_m^2(2m)!\sup_n \frac{1}{n^2}\sum_{i,j\geq 1, i\neq j}^{n-1} R_{H_1}\left(1,\left(\frac{j}{i}\right)\right)^{2m}\left(\frac{j}{i}\right)^{-2H_1 m}\left(\frac{i}{n}\frac{j}{n}\right)^{-H_1}.
\end{aligned}$$

Let's focus on the case where $H_1<\frac{1}{2}$ first. Let $Q_{H_1}(z)$ be the function defined by

$$Q_{H_1}(z)=\begin{cases}\frac{R_{H_1}(1,z)}{z^{H_1}} & \text{if} \quad z\in(0,1]\\ 0 \quad \text{if} & z=0.\end{cases}$$

For $H_1<\frac{1}{2}$, we have

$$Q_{H_1}(z)\leq z^{H_1}.$$

Indeed, the function $f(z)=1-z^{2H_1}-(1-z)^{2H_1}$ is negative on $[0,1]$, increasing on $\left[\frac{1}{2},1\right]$, decreasing on $\left[0,\frac{1}{2}\right]$ and $f(1)=f(0)=0$. It follows that

$$\begin{aligned}
&\sum_{m\geq 0} C_m^2(2m)!\sup_{n,\varepsilon}|A_m(\varepsilon,n)|\\
&\leq 2\sum_{m\geq 0} C_m^2(2m)!\sup_n \frac{1}{n^2}\sum_{i,j=0;i>j}^{n-1}\left(\frac{j}{i}\right)^{2H_1 m}\left(\frac{i}{n}\frac{j}{n}\right)^{-H_1}\\
&\leq 2\sum_{m\geq 0} C_m^2(2m)!\sup_n \frac{1}{n^2}\sum_{i,j=0;i>j}^{n-1}\left(\frac{j}{n}\right)^{H_1(2m-1)}\left(\frac{i}{n}\right)^{-H_1(2m+1)}\\
&\leq 2\sum_{m\geq 0} C_m^2(2m)!\sup_n n^{2H_1-2}\sum_{i=0}^{n-1} i^{-H_1(2m+1)}\sum_{j=1}^{i-1}\int_j^{j+1} j^{H_1(2m-1)}dx\\
&\leq 2\sum_{m\geq 0} C_m^2(2m)!\sup_n n^{2H_1-2}\sum_{i=0}^{n-1} i^{-H_1(2m+1)}\int_0^i x^{H_1(2m-1)}dx\\
&\leq 2\sum_{m\geq 0} C_m^2(2m)!\sup_n \frac{n^{2H_1-2}}{2H_1m-H_1+1}\sum_{i=0}^{n-1} i^{1-2H_1}\\
&\leq 2\sum_{m\geq 0} C_m^2(2m)!\sup_n \frac{n^{-1}}{2H_1m-H_1+1}\sum_{i=0}^{n-1} 1\\
&\leq 2\sum_{m\geq 0} C_m^2(2m)!\sup_n \frac{1}{2H_1m-H_1+1}\leq 2\sum_{m\geq 0}\frac{C_m^2(2m)!}{2H_1m-H_1+1}.
\end{aligned}$$

Given that, by using Stirling's formula, the coefficient $C_m^2(2m)!$ behaves as $\frac{1}{\sqrt{m}}$, we obtain that the above sum is finite. Thus, we obtain the convergence of $a_n^{(3)}$ to zero in $L^2(\Omega)$ for $H_1<\frac{1}{2}$.

Let us now treat the case $H_1 > \frac{1}{2}$. We know (see [ELS+05], Lemma 1) that the function Q_H is increasing on $[0,1]$. Since $\frac{j}{i} \leq \frac{i-1}{i} = 1 - \frac{1}{i}$ it holds that $Q_H(\frac{j}{i}) \leq Q_H(1-\frac{1}{i})$. Then

$$\begin{aligned}
&\sum_{m\geq 0} C_m^2 (2m)! \sup_{n,\varepsilon} |A_m(\varepsilon, n)| \\
&\leq 2 \sum_{m\geq 0} C_m^2 (2m)! \sup_n \frac{1}{n^2} \sum_{i=1}^{n-1} Q_H\left(1-\frac{1}{i}\right) \sum_{j=1}^{n-1} \left(\frac{i}{n}\frac{j}{n}\right)^{-H_1} \\
&= 2 \sum_{m\geq 0} C_m^2 (2m)! \sup_n \frac{1}{n} \sum_{i=1}^{n-1} Q_H\left(1-\frac{1}{i}\right) \left(\frac{i}{n}\right)^{-H_1} \sum_{j=1}^{i-1} \int_{\frac{j-1}{n}}^{\frac{1}{n}} x^{-H_1} dx \\
&\leq c_H \sum_{m\geq 0} C_m^2 (2m)! \sup_n \frac{1}{n} \sum_{i=1}^{n-1} Q_H\left(1-\frac{1}{i}\right) \left(\frac{i}{n}\right)^{-H_1} \left(\frac{i-1}{n}\right)^{1-H_1} \\
&\sim c_H \sum_{m\geq 0} C_m^2 (2m)! \sup_n \frac{1}{n} \sum_{i=1}^{n-1} Q_H\left(1-\frac{1}{i}\right) \left(\frac{i}{n}\right)^{1-2H_1}.
\end{aligned}$$

By adapting Lemma 2 in [ELS+05] (by separating the sum over i in a sum with $\frac{1}{i} \leq \delta$ and $\frac{1}{i} > \delta$ with δ suitably chosen), we can prove that

$$\frac{1}{n} \sum_{i,j=0}^{n-1} R_{H_1}\left(1, \left(\frac{j}{i}\right)\right)^{2m} \left(\frac{j}{i}\right)^{-2H_1 m} \left(\frac{i}{n}\frac{j}{n}\right)^{-H} \leq c(H_1) m^{-\frac{1}{2H_1}}$$

with $c(H_1)$ not depending on m nor n. As a consequence

$$\sum_{m\geq 0} c_m^2 (2m)! \sup_{n,\varepsilon} |A_m(\varepsilon, n)| \leq c(H_1) c_m^2 (2m)! m^{-\frac{1}{2H_1}}.$$

The Stirling formula implies again that the above series is finite. ■

Theorem 1. *Let $\langle S\rangle_n$ be given by (1.3). Then, as $n \to \infty$, we have the convergence in distribution*

$$n^{\alpha+H_1-1}\langle S\rangle_n \to \int_{\mathbb{R}} K^2(y) dy L^{H_1}(1,0)$$

where $L^{H_1}(1,0)$ is the local time of the fractional Brownian motion B^{H_1}.

Proof : Using Proposition 4 it suffices to check that $n^{\alpha+H_1}\int_0^1 K^2(n^{\alpha+H_1}B_s^{H_1})ds$ converges to $\int_{\mathbb{R}} K^2(y)dyL^{H_1}(1,0)$. Using the occupation time formula (1.17), we obtain

$$\begin{aligned}
n^{\alpha+H_1} \int_0^1 K^2(n^{\alpha+H_1} B_s^{H_1}) ds &= n^{\alpha+H_1} \int_{\mathbb{R}} K^2(n^{\alpha+H_1} x) L^{H_1}(1,x) dx \\
&= \int_{\mathbb{R}} K^2(y) L(1, yn^{-\alpha-H_1}) dy
\end{aligned}$$

which converges as $n \to \infty$ to $\int_{\mathbb{R}} K^2(y)dyL^{H_1}(1,0)$ by using the continuity properties of the local time. ■

1.5 Limit distribution of S_n

In this paragraph, we prove the limit in distribution of (2.7). Recall the notation (1.13) and let's consider the Gaussian vector

$$X^{H_2} = (X_1^{H_2}, ..., X_n^{H_2}) = (B_1^{H_2} - B_0^{H_2}, ..., B_n^{H_2} - B_{n-1}^{H_2}).$$

From this definition, it follows that

$$S_n = \sum_{i=0}^{n-1} K(n^\alpha B_i^{H_1})(B_{i+1}^{H_2} - B_i^{H_2}) = \sum_{i=0}^{n-1} K(n^\alpha B_i^{H_1}) X_{i+1}^{H_2}.$$

Theorem 2. *Let (S_n) be given by (2.7) and assume that*

$$\alpha < 1 - H_1 \tag{1.26}$$

Then we have the convergence in law

$$n^{\alpha+H_1-1} S_n \underset{n\to+\infty}{\longrightarrow} d_1 W_{L^{H_1}(1,0)}$$

where $L^{H_1}(1,0)$ is the local time of B^{H_1}, $d_1 := \int_{\mathbb{R}} K^2(y)dy$ and W is a Brownian motion independent from B^{H_1}.

Proof : We will study the characteristic function of $n^{\frac{\alpha}{2}+\frac{H_1}{2}-\frac{1}{2}} S_n$. In order to simplify the presentation, we will use the following notation. Let i_0 be the imaginary unit and λ_n be

$$\lambda_n = \lambda n^{\frac{\alpha}{2}+\frac{H_1}{2}-\frac{1}{2}} \quad \text{with} \quad \lambda \in \mathbb{R}.$$

Using the independence of the two fBms and computing the conditional expectation of $e^{i\lambda_n S_n}$ given B^{H_1} we get

$$\mathbf{E}\left(e^{i_0\lambda_n S_n}\right) = \mathbf{E}\left(e^{-\frac{1}{2}\sum_{i,j=0}^{n-1} \lambda_n^2 K\left(n^\alpha B_i^{H_1}\right) K\left(n^\alpha B_j^{H_1}\right) f_{H_2}(i,j)}\right)$$

because if X is a Gaussian vector with mean μ and covariance matrix Σ, it's characteristic function is given by

$$\mathbf{E}\left(e^{i_0\langle t,X\rangle}\right) = e^{i_0\mu^{\mathrm{T}} t - \frac{1}{2} t^{\mathrm{T}} \Sigma t}.$$

It follows that, with $f_{H_2}(i,j)$ given by (1.13),

$$\begin{aligned}
&\mathbf{E}\left(e^{i_0\lambda_n S_n}\right) \\
&= \mathbf{E}\left(e^{-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right)} e^{-\frac{\lambda_n^2}{2}\sum_{i\neq j=0}^{n-1} K\left(n^\alpha B_i^{H_1}\right) K\left(n^\alpha B_j^{H_1}\right) f_{H_2}(i,j)}\right) \\
&= \mathbf{E}\left(e^{-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right)} e^{-\lambda_n^2\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right) K\left(n^\alpha B_j^{H_1}\right) f_{H_2}(i,j)}\right) \\
&= \mathbf{E}\left(e^{-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right)}\right. \\
&\left.\times\, e^{-\lambda_n^2\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right) K\left(n^\alpha B_j^{H_1}\right) H_2(2H_2-1)\int_i^{i+1}\int_j^{j+1} |s-u|^{2H_2-2} du ds}\right) \\
&= \mathbf{E}\left(e^{-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right)} e^{-\lambda_n^2 H_2(2H_2-1)\int_0^n\int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right) K\left(n^\alpha B_{[u]}^{H_1}\right) |s-u|^{2H_2-2} du ds}\right).
\end{aligned}$$

Consider the process $(V_n)_{n\geq 0}$ defined by

$$V_n = \int_0^n \int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right) K\left(n^\alpha B_{[u]}^{H_1}\right) |s-u|^{2H_2-2} \, duds$$

and the function ψ defined by $\psi(x) = e^{-\lambda_n^2 H_2(2H_2-1)x}$. Note that, since we excluded the diagonal, the integral $duds$ in the expression of V_n makes sense even for $H_2 < \frac{1}{2}$. Note also that V_n is a bounded variation process (its quadratic variation is 0). Furthermore,

$$\psi'(x) = -\lambda_n^2 H_2(2H_2-1)e^{-\lambda_n^2 H_2(2H_2-1)x}.$$

Using the change of variables formula for bounded variation processes, it follows that

$$\psi(V_n) = 1 + \int_0^n \psi'(V_s)dV_s$$

i.e.,

$$e^{-\lambda_n^2 H_2(2H_2-1)V_n} = 1 - \lambda_n^2 H_2(2H_2-1)\int_0^n e^{-\lambda_n^2 H_2(2H_2-1)V_s} dV_s.$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left(e^{i_0\lambda_n S_n}\right) &= \mathbf{E}\left(e^{-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right)}\left(1-\lambda_n^2 H_2(2H_2-1)\int_0^n e^{-\lambda_n^2 H_2(2H_2-1)V_s} dV_s\right)\right)\\
&= \mathbf{E}\left(e^{-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right)}\right)\\
&\quad -\mathbf{E}\left(\lambda_n^2 H_2(2H_2-1)e^{-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right)}\int_0^n e^{-\lambda_n^2 H_2(2H_2-1)V_s} dV_s\right)\\
&:= \mathbf{E}\left(T_1\right) - \mathbf{E}\left(T_2\right).
\end{aligned}$$

We will now focus on the term $\mathbf{E}\left(T_2\right)$ and show that

$$T_2 \xrightarrow{L^1} 0.$$

From

$$dV_s = \left(\int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right) K\left(n^\alpha B_{[u]}^{H_1}\right) |s-u|^{2H_2-2} \, du\right) ds$$

we get

$$\begin{aligned}
\mathbf{E}\left(T_2\right) &= \mathbf{E}\Bigg(\lambda_n^2 H_2(2H_2-1)e^{-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right)}\\
&\quad \times \int_0^n e^{-\lambda_n^2 H_2(2H_2-1)V_s} \int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right) K\left(n^\alpha B_{[u]}^{H_1}\right) |s-u|^{2H_2-2} \, duds\Bigg)\\
&= \mathbf{E}\Bigg(\lambda_n^2 H_2(2H_2-1)\int_0^n e^{-\frac{\lambda_n^2}{2}\int_0^s K^2\left(n^\alpha B_{[u]}^{H_1}\right)du} e^{-\frac{\lambda_n^2}{2}\int_s^n K^2\left(n^\alpha B_{[u]}^{H_1}\right)du}\\
&\quad \times e^{-\lambda_n^2 H_2(2H_2-1)V_s}\int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right) K\left(n^\alpha B_{[u]}^{H_1}\right) |s-u|^{2H_2-2} \, duds\Bigg).
\end{aligned}$$

Recall that the following holds

$$\mathbf{E}\left(e^{i_0\lambda_n S_s}|B_s^{H_1}\right) = \mathbf{E}\left(e^{-\frac{\lambda_n^2}{2}\int_0^s K^2\left(n^\alpha B_{[u]}^{H_1}\right)du} e^{-\lambda_n^2 H_2(2H_2-1)V_s}|B_s^{H_1}\right). \tag{1.27}$$

This can be seen for s integer as at the beginning of this proof and also (1.27) can easily be checked for any $s>0$. We will use this property to compute the following upper bound for $\mathbf{E}(|T_2|)$

$$\begin{aligned}
\mathbf{E}(|T_2|) &\le \mathbf{E}\left(\lambda_n^2\int_0^n e^{-\frac{\lambda_n^2}{2}\int_0^s K^2\left(n^\alpha B_{[u]}^{H_1}\right)du} e^{-\lambda_n^2 H_2(2H_2-1)V_s} \underbrace{\left|e^{-\frac{\lambda_n^2}{2}\int_s^n K^2\left(n^\alpha B_{[u]}^{H_1}\right)du}\right|}_{\le 1}\right. \\
&\qquad \left.\times\int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right)K\left(n^\alpha B_{[u]}^{H_1}\right)H_2\,|2H_2-1|\,|s-u|^{2H_2-2}\,duds\right) \\
&\le \mathbf{E}\left(\lambda_n^2\int_0^n \mathbf{E}\left(e^{-\frac{\lambda_n^2}{2}\int_0^s K^2\left(n^\alpha B_{[u]}^{H_1}\right)du} e^{-\lambda_n^2 H_2(2H_2-1)V_s}|B_s^{H_1}\right)\right. \\
&\qquad \left.\times\int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right)K\left(n^\alpha B_{[u]}^{H_1}\right)H_2\,|2H_2-1|\,|s-u|^{2H_2-2}\,duds\right).
\end{aligned}$$

This is true because all the terms of the double integral are measurable with respect to the filtration generated by $(B_u^{H_1}, u\le s)$. At this point, we use (1.27) to write

$$\begin{aligned}
&\mathbf{E}(|T_2|) \\
&\le \mathbf{E}\left(\lambda_n^2\int_0^n \mathbf{E}\left(e^{i_0\lambda_n S_s}|B_s^{H_1}\right)\int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right)K\left(n^\alpha B_{[u]}^{H_1}\right)H_2\,|2H_2-1|\,|s-u|^{2H_2-2}\,duds\right) \\
&\le \mathbf{E}\left(\lambda_n^2\int_0^n \underbrace{\left|e^{i_0\lambda_n S_s}\right|}_{=1}\int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right)K\left(n^\alpha B_{[u]}^{H_1}\right)H_2\,|2H_2-1|\,|s-u|^{2H_2-2}\,duds\right) \\
&\le \mathbf{E}\left(\lambda_n^2\int_0^n\int_0^{[s]} K\left(n^\alpha B_{[s]}^{H_1}\right)K\left(n^\alpha B_{[u]}^{H_1}\right)H_2\,|2H_2-1|\,|s-u|^{2H_2-2}\,duds\right) \\
&\le \mathbf{E}\left(\lambda_n^2\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right)K\left(n^\alpha B_j^{H_1}\right)H_2\,|2H_2-1|\int_i^{i+1}\int_j^{j+1}|s-u|^{2H_2-2}\,duds\right).
\end{aligned}$$

Assume that $H_2>\frac{1}{2}$, ergo $|2H_2-1|>0$ and $f_{H_2}(i,j)>0$. Consequently,

$$\begin{aligned}
\mathbf{E}(|T_2|) &\le \mathbf{E}\left(\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right)K\left(n^\alpha B_j^{H_1}\right)f_{H_2}(i,j)\right) \\
&\le \mathbf{E}\left(\frac{\lambda^2}{2}n^{\alpha+H_1-1}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right)K\left(n^\alpha B_j^{H_1}\right)f_{H_2}(i,j)\right).
\end{aligned}$$

The previous term is exactly the non-diagonal term of the L^2-norm of $n^{\frac{\alpha}{2}+\frac{H_1}{2}-\frac{1}{2}}S_n$ and we know that under condition (1.11), it converges to zero when $n\to+\infty$. Finally we have

$$\mathbf{E}(|T_2|) \underset{n\to+\infty}{\longrightarrow} 0.$$

Assume now that $H_2 < \frac{1}{2}$. It follows that $|2H_2 - 1| < 0$ and $f_{H_2}(i,j) < 0$, which gives us

$$\begin{aligned}
\mathbf{E}\left(|T_2|\right) &\leq \mathbf{E}\left(-\frac{\lambda_n^2}{2}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right) K\left(n^\alpha B_j^{H_1}\right) f_{H_2}(i,j)\right) \\
&\leq \mathbf{E}\left(-\frac{\lambda^2}{2} n^{\alpha+H_1-1}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right) K\left(n^\alpha B_j^{H_1}\right) f_{H_2}(i,j)\right).
\end{aligned}$$

As in the previous case, this term is again exactly the non-diagonal term of the L^2-norm of $n^{\frac{\alpha}{2}+\frac{H_1}{2}-\frac{1}{2}} S_n$ and for the same reasons, we get the following result again (which is now valid for any $H_2 \in (0,1)$)

$$\mathbf{E}\left(|T_2|\right) \underset{n\to+\infty}{\longrightarrow} 0.$$

Concerning the term T_1, we note that

$$\mathbf{E}\left(T_1\right) = \mathbf{E}\left(e^{-\frac{\lambda^2}{2}\langle S\rangle_n}\right)$$

and the result follows from Theorem 1. ■

Remark 1. *The following comments deal with the conditions (1.11) and (1.26). Condition (1.26) is a natural extension of the condition $\alpha < \frac{1}{2}$ in e.g. [WP09a], [WP09b] which means that the bandwidth parameter satisfies $nh_n^2 = nn^{-2\alpha} \to \infty$ as $n \to \infty$. From (1.11) and (1.26), this is the constraint we find for α (considering α is our degree of freedom)*

$$\left\{\begin{array}{r} 0 < H_1 < 1 \\ 0 < H_2 < 1 \\ \alpha > 4H_2 - H_1 - 2 \\ \alpha < 1 - H_1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{r} 0 < H_1 < 1 \\ 0 < H_2 < 1 \\ 4H_2 - H_1 - 2 < \alpha < 1 - H_1. \end{array}\right.$$

As an example, consider the case where $H_1 = H_2 = H$. Those constraints become

$$\left\{\begin{array}{r} 0 < H < 1 \\ (3H-2)^+ < \alpha < 1 - H. \end{array}\right.$$

For this system to have a solution, we need to verify that

$$3H - 2 < 1 - H \Leftrightarrow H < \frac{3}{4}.$$

As a result, our constraints become

$$\left\{\begin{array}{r} 0 < H < \frac{3}{4} \\ (3H-2)^+ < \alpha < 1 - H. \end{array}\right.$$

We could also consider the case where α has a fixed value and where the constraints would be on H_1 and H_2.

1.6 The stable convergence

In this section we will study the convergence of the vector $(S_n, (G_t)_{t\geq 0})$ where $(G_t)_{t\geq 0}$ is a stochastic process independent from B^{H_1} and satisfies some additional conditions. In this case, since the process $(G_t)_{t\geq 0}$ is not necessarily a Gaussian process and since no information is available on the correlation between B^{H_2} and G_t, the characteristic function of the vector $(S_n, (G_t)_{t\geq 0})$ cannot be computed directly. To compute it, we will use the tools of the stochastic calculus with respect to the fractional Brownian motion. The basic observation is that S_n can be expressed as a stochastic integral with respect to B^{H_2}. Indeed,

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} K(n^\alpha B_i^{H_1})(B_{i+1}^{H_2} - B_i^{H_2}) = \sum_{i=0}^{n-1} K(n^\alpha B_i^{H_1})\delta^{H_2}(\mathbf{1}_{[i,i+1]}(\cdot)) \qquad (1.28)\\ &= \sum_{i=0}^{n-1} \delta^{H_2}(K(n^\alpha B_i^{H_1})\mathbf{1}_{[i,i+1]}(\cdot)) + \left\langle \underbrace{D^{H_2}K(n^\alpha B_i^{H_1})}_{=0 \text{ from } B^{H_1}\perp B_t^{H_2}}, \mathbf{1}_{[i,i+1]}(\cdot)\right\rangle_{\mathcal{H}_{H_2}} \\ &= \sum_{i=0}^{n-1} \int_i^{i+1} K(n^\alpha B_i^{H_1})dB_s^{H_2} = \sum_{i=0}^{n-1} \int_i^{i+1} K(n^\alpha B_{[s]}^{H_1})dB_s^{H_2} = \int_0^n K(n^\alpha B_{[s]}^{H_1})dB_s^{H_2}. \end{aligned}$$

We will also use the "bracket" of S_n. This quantity equals

$$\begin{aligned} \sum_{i=0}^{n-1} K^2(n^\alpha B_i^{H_1}) &= \sum_{i=0}^{n-1} \int_i^{i+1} K^2(n^\alpha B_i^{H_1})ds \\ &= \sum_{i=0}^{n-1} \int_i^{i+1} K^2(n^\alpha B_{[s]}^{H_1})ds = \int_0^n K^2(n^\alpha B_{[s]}^{H_1})ds. \end{aligned}$$

Before going any further, we will describe the elements of the stochastic calculus with respect to fractional Brownian motion that we will be using in the sequel. We will start by introducing some notations and definitions. Let ϕ be the function defined by

$$\phi(s,t) = H(2H-1)\,|s-t|^{2H-2}, \quad s,t \in \mathbb{R}.$$

Let D (introduced in section 2) be the Malliavin derivative operator with respect to the fractional Brownian motion with Hurst parameter H. Based on this operator, let D^ϕ be another derivative operator (called the ϕ-derivative operator) defined by

$$D_t^\phi F = \int_{\mathbb{R}} \phi(t,v) D_v F dv$$

for any F in the domain of D. For more details about this operator, see [BHØZ08]. Let $\mathcal{L}_\phi(0,T)$ be the family of stochastic processes F on $[0,T]$ with the following properties : $F \in \mathcal{L}_\phi(0,T)$ if and only if $\mathbf{E}\left[\|F\|_{\mathcal{H}}^2\right] < \infty$, F is ϕ-differentiable, the trace of $D_s^\phi F_t, 0 \leq s,t \leq T$, exists, and $\mathbf{E}\left[\int_0^T\int_0^T \left|D_s^\phi F_t\right|^2 dsdt\right] < \infty$ and for each sequence of partitions $(\pi_n, n\in\mathbb{N})$ such that $|\pi_n| \to 0$ as $n\to+\infty$,

$$\sum_{i,j=0}^{n-1} \mathbf{E}\left[\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \left|D_s^\phi F_{t_i^{(n)}}^\pi D_t^\phi F_{t_j^{(n)}}^\pi - D_s^\phi F_t D_t^\phi F_s\right| dsdt\right]$$

and

$$\mathbf{E}\left[\|F^{\pi}-F\|_{\mathcal{H}}^{2}\right]$$

tend to 0 as $n \to +\infty$, where $\pi_n :< 0 = t_0^{(n)} < t_1^{(n)} < ... < t_{n-1}^{(n)} < t_n^{(n)} = T$.

In our particular situation, we are dealing with processes of the form $\int_0^t F_u dB_u^H + \int_0^t G_u du$, (where B^H is a fractional Brownian motion with Hurst parameter H), for which the following Itô formula holds in the case $H > \frac{1}{2}$.

Theorem 3. *Let $\eta_t = \int_0^t F_u dB_u^H + \int_0^t G_u du$, for $t \in [0,T]$ with $\mathbf{E}\left[\underset{0\le s\le T}{sup}|G_s|\right] < \infty$ and let $(F_u, 0 \le u \le T)$ be a stochastic process in $\mathcal{L}_\phi(0,T)$. Assume that there is a $\beta > 1-H$ such that*

$$\mathbf{E}\left[|F_u - F_v|^2\right] \le C\,|u-v|^{2\beta} \tag{1.29}$$

where $|u-v| \le \zeta$ for some $\zeta > 0$ and

$$\underset{0\le u,v\le t,|u-v|\to 0}{lim}\mathbf{E}\left[\left|D_u^\phi(F_u - F_v)\right|^2\right] = 0. \tag{1.30}$$

Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function having the first continous derivative in its first variable and the second continous derivative in its second variable. Assume that these derivatives are bounded. Moreover, it is assumed that $\mathbf{E}\left[\int_0^T \left|F_s D_s^\phi \eta_s\right| ds\right] < \infty$ and $(\frac{\partial f(s,\eta_s)}{\partial x}F_s, s \in [0,T]) \in \mathcal{L}_\phi(0,T)$. Then for $t \in [0,T]$,

$$\begin{aligned} f(t,\eta_t) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,\eta_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,\eta_s)G_s ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s,\eta_s)F_s dB_s^H + \int_0^t \frac{\partial^2 f}{\partial x^2}(s,\eta_s)F_s D_s^\phi \eta_s ds. \end{aligned} \tag{1.31}$$

We also have the following technical lemma ([BHØZ08] p.71) that will be particularly useful for our future computations.

Lemma 4. *Let $(F_t, t \in [0,T])$ be a stochastic process in $\mathcal{L}_\phi(0,T)$ and*

$$\underset{0\le s\le T}{sup}\mathbf{E}\left[\left|D_s^\phi F_s\right|^2\right] < \infty$$

and let $\eta_t = \int_0^t F_u dB_u^H$ for $t \in [0,T]$. Then, for $s,t \in [0,T]$,

$$D_s^\phi \eta_t = \int_0^t D_s^\phi F_u dB_u^H + \int_0^t F_u \phi(s,u)du. \tag{1.32}$$

It is now possible to state the main result of this section

Theorem 4. *Assume that (1.11) and (1.26) holds. Let $(G_t)_{t\ge 0}$ be a stochastic process independent from B^{H_1} and adapted to the filtration generated by B^{H_2} such that for every $t \ge 0$ the random variable G_t belongs to $\mathbb{D}^{1,2}$ and $\|D_s G_t| \le C$ for any s,t and ω. Then the vector $(S_n, (G_t)_{t\ge 0})$ converges in the sense of finite dimensional distributions to the vector $(cW_{L^{H_1}(1,0)}, (G_t)_{t\ge 0}$, where c is a positive constant.*

Proof : In order to simplify the presentation, the following notations will be used. We will denote by λ_n (like we did in a previous proof) the quantity

$$\lambda_n = \lambda n^{\frac{\alpha}{2}+\frac{H_1}{2}-\frac{1}{2}}$$

where $\lambda \in \mathbb{R}$. The following notation will also be used :

$$e(\lambda, n) = e^{-\frac{\lambda^2}{2}n^{\alpha+H_1-1}\int_0^n K^2(n^\alpha B^{H_1}_{[u]})du} = e^{-\frac{\lambda^2}{2}n^{\alpha+H_1-1}\sum_{i=0}^{n-1}K^2(n^\alpha B^{H_1}_i)}.$$

Let $(F_t, t \geq 0)$ and $(G_t, t \geq 0)$ be two stochastic processes defined by

$$\begin{cases} F_u = K(n^\alpha B^{H_1}_{[u]}) \\ G_u = -i_0\frac{\lambda_n}{2}K^2(n^\alpha B^{H_1}_{[u]}) \end{cases}$$

and let $(\eta_t^{(\lambda_n)}, t \geq 0)$ be the stochastic process defined by

$$\eta_t^{(\lambda_n)} = \int_0^t F_u dB^H_u + \int_0^t G_u du = \int_0^t K(n^\alpha B^{H_1}_{[u]})dB^{H_2}_u - i_0\frac{\lambda_n}{2}\int_0^t K^2(n^\alpha B^{H_1}_{[u]})du.$$

Consider the function $f : \mathbb{C} \to \mathbb{C}, f(x) = e^{i_0\lambda_n x}$. We can apply the Itô formula to $f(\eta_t^{(\lambda_n)})$ in order to obtain

$$\begin{aligned} e^{i_0\lambda_n\eta_t^{(\lambda_n)}} &= 1 + \frac{\lambda_n^2}{2}\int_0^t e^{i_0\lambda_n\eta_s^{(\lambda_n)}}K^2(n^\alpha B^{H_1}_{[s]})ds \\ &\quad + i_0\lambda_n\int_0^t e^{i_0\lambda_n\eta_s^{(\lambda_n)}}K(n^\alpha B^{H_1}_{[s]})dB^{H_2}_s \\ &\quad -\lambda_n^2\int_0^t e^{i_0\lambda_n\eta_s^{(\lambda_n)}}K(n^\alpha B^{H_1}_{[s]})D^{\phi,H_2}_s\eta_s^{(\lambda_n)}ds \end{aligned} \tag{1.33}$$

where D^{ϕ,H_2} is the operator D^ϕ introduced above with respect to the fractional Brownian motion B^{H_2}. We use Lemma 4 to compute $D^{\phi,H_2}_s\eta_s^{(\lambda_n)}$. We get

$$\begin{aligned} D^{\phi,H_2}_s\eta_s^{(\lambda_n)} &= D^{\phi,H_2}_s\int_0^s K(n^\alpha B^{H_1}_{[u]})dB^{H_2}_u - i_0\frac{\lambda_n}{2}\underbrace{D^{\phi,H_2}_s\int_0^s K^2(n^\alpha B^{H_1}_{[u]})du}_{=0 \text{ from } B^{H_1}\perp B^{H_2}_t} \\ &= \underbrace{\int_0^s D^{\phi,H_2}_s K(n^\alpha B^{H_1}_{[u]})dB^{H_2}_u}_{=0 \text{ from } B^{H_1}\perp B^{H_2}_t} + H_2(2H_2-1)\int_0^s K(n^\alpha B^{H_1}_{[u]})\,|s-u|^{2H_2-2}\,du \\ &= H_2(2H_2-1)\int_0^s K(n^\alpha B^{H_1}_{[u]})\,|s-u|^{2H_2-2}\,du. \end{aligned}$$

By substituting in (1.33), we obtain

$$\begin{aligned} e^{i_0\lambda_n\eta_t^{(\lambda_n)}} &= 1 + \frac{\lambda_n^2}{2}\int_0^t e^{i_0\lambda_n\eta_s^{(\lambda_n)}}K^2(n^\alpha B^{H_1}_{[s]})ds \\ &\quad + i_0\lambda_n\int_0^t e^{i_0\lambda_n\eta_s^{(\lambda_n)}}K(n^\alpha B^{H_1}_{[s]})dB^{H_2}_s \\ &\quad -\lambda_n^2 H_2(2H_2-1)\int_0^t e^{i_0\lambda_n\eta_s^{(\lambda_n)}}K(n^\alpha B^{H_1}_{[s]}) \\ &\quad \times\int_0^s K(n^\alpha B^{H_1}_{[u]})\,|s-u|^{2H_2-2}\,duds. \end{aligned} \tag{1.34}$$

By multiplying both sides of (1.34) by $e^{-\frac{\lambda_n^2}{2}\int_0^n K^2(n^\alpha B^{H_1}_{[u]})du} = e(\lambda, n)$, we obtain

$$
\begin{aligned}
& e^{i_0\lambda_n \int_0^n K(n^\alpha B^{H_1}_{[u]})dB^{H_2}_u} = e^{-\frac{\lambda_n^2}{2}\int_0^n K^2(n^\alpha B^{H_1}_{[u]})du} \\
& +\frac{\lambda_n^2}{2}\int_0^n e^{i_0\lambda_n \eta_s^{(\lambda_n)}} K^2(n^\alpha B^{H_1}_{[s]})ds \cdot e(\lambda,n) \\
& +i_0\lambda_n \int_0^n e^{i_0\lambda_n \eta_s^{(\lambda_n)}} K(n^\alpha B^{H_1}_{[s]})dB^{H_2}_s \cdot e(\lambda,n) \\
& -\lambda_n^2 H_2(2H_2-1)\int_0^n e^{i_0\lambda_n \eta_s^{(\lambda_n)}} K(n^\alpha B^{H_1}_{[s]}) \\
& \times \int_0^s K(n^\alpha B^{H_1}_{[u]})\,|s-u|^{2H_2-2}\,duds \cdot e(\lambda,n). \qquad (1.35)
\end{aligned}
$$

The sum of the last two terms in (1.35) can be written in a more suitable way by using sums instead of integrals. Together, these two last terms give us

$$
\begin{aligned}
& \mathbf{E}\Bigg(\lambda_n^2 \int_0^n e^{i_0\lambda_n \eta_s^{(\lambda_n)}} K\left(n^\alpha B^{H_1}_{[s]}\right) \\
& \times \left[\frac{1}{2}K\left(n^\alpha B^{H_1}_{[s]}\right) - H_2(2H_2-1)\int_0^s K\left(n^\alpha B^{H_1}_{[u]}\right)|s-u|^{2H_2-2}\,du\right] ds \cdot e(\lambda,n)\Bigg) \\
= \;& \mathbf{E}\Bigg(\lambda_n^2 \int_0^n e^{i_0\lambda_n \eta_s^{(\lambda_n)}} K\left(n^\alpha B^{H_1}_{[s]}\right) \\
& \times \left[\frac{1}{2}K\left(n^\alpha B^{H_1}_{[s]}\right) - H_2(2H_2-1)\left(\int_0^i K\left(n^\alpha B^{H_1}_{[u]}\right)|s-u|^{2H_2-2}\,du\right.\right. \\
& \left.\left. + \int_i^s K\left(n^\alpha B^{H_1}_{i}\right)|s-u|^{2H_2-2}\,du\right)\right] ds \cdot e(\lambda,n)\Bigg) \\
= \;& -\mathbf{E}\Bigg(\lambda_n^2 \sum_{i=0}^{n-1}\int_i^{i+1} e^{i_0\lambda_n \eta_s^{(\lambda_n)}} K\left(n^\alpha B^{H_1}_{i}\right) \\
& \times H_2(2H_2-1)\sum_{j=0}^{i-1} K\left(n^\alpha B^{H_1}_{j}\right)\int_j^{j+1}|s-u|^{2H_2-2}\,duds \cdot e(\lambda,n)\Bigg) \\
& +\mathbf{E}\Bigg(\lambda_n^2 \sum_{i=0}^{n-1}\int_i^{i+1} e^{i_0\lambda_n \eta_s^{(\lambda_n)}} K^2\left(n^\alpha B^{H_1}_{i}\right) \\
& \times \left[\frac{1}{2} - H_2(2H_2-1)\int_i^s |s-u|^{2H_2-2}\,du\right] ds \cdot e(\lambda,n)\Bigg). \qquad (1.36)
\end{aligned}
$$

Let us now fix $\beta_1, \ldots, \beta_N \in \mathbb{R}$ and $t_1, \ldots, t_N \geq 0$. We need to show that

$$
\mathbf{E}\left(e^{i_0\lambda_n S_n} e^{i_0(\beta_1 G_{t_1}+\ldots+\beta_N G_{t_N})}\right)
$$

converges to $\mathbf{E}\left(e^{-\frac{\lambda^2 (L^{H_1}(1,0))^2}{2}} e^{i_0(\beta_1 G_{t_1}+\ldots+\beta_N G_{t_N})}\right)$. We will use the notation

$$
g_N := e^{i_0(\beta_1 G_{t_1}+\ldots+\beta_N G_{t_N})}.
$$

By combining relations (1.35) and (1.36), we can write

$$\begin{aligned}
\mathbf{E}\left(e^{i_0\lambda_n S_n} g_N\right) &= \mathbf{E}(e(\lambda,n)g_N) \\
&+\mathbf{E}\left(i_0\lambda_n \int_0^n e^{i_0\lambda_n\eta_s^{(\lambda_n)}} K(n^\alpha B_{[s]}^{H_1}) dB_s^{H_2}\cdot e(\lambda,n)g_N\right) \\
&-\mathbf{E}\left(\lambda_n^2 \sum_{i=0}^{n-1}\int_i^{i+1} e^{i_0\lambda_n\eta_s^{(\lambda_n)}} K\left(n^\alpha B_i^{H_1}\right)\right. \\
&\left.\times H_2(2H_2-1)\sum_{j=0}^{i-1} K\left(n^\alpha B_j^{H_1}\right)\int_j^{j+1}|s-u|^{2H_2-2}\,duds\times e(\lambda,n)g_N\right) \\
&+\mathbf{E}\left(\lambda_n^2 \sum_{i=0}^{n-1}\int_i^{i+1} e^{i_0\lambda_n\eta_s^{(\lambda_n)}} K^2\left(n^\alpha B_i^{H_1}\right)\right. \\
&\left.\times\left[\frac{1}{2}-H_2(2H_2-1)\int_i^s|s-u|^{2H_2-2}\,du\right]ds\times e(\lambda,n)g_N\right) \\
&:= \mathbf{E}(e(\lambda,n)g_N)+T_1^\star+T_2^\star+T_3^\star. \qquad (1.37)
\end{aligned}$$

Let us begin by proving that the term $T_2^\star$ converges to zero as $n\to\infty$. Since

$$\left|e^{i_0\lambda_n\eta_u^{(\lambda_n)}}\right| e^{-\frac{\lambda_n^2}{2}\int_0^n K^2\left(n^\alpha B_{[u]}^{H_1}\right)du} \leq 1 \qquad (1.38)$$

for every $s\leq n$ and since $|e^{i_0x}|=1$ for every x real, $T_2^\star$ can be bounded as follows

$$\begin{aligned}
T_2^\star &\leq \mathbf{E}\left(\lambda_n^2 c(H_2)\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right)K\left(n^\alpha B_j^{H_1}\right)\int_i^{i+1}\int_j^{j+1}|s-u|^{2H_2-2}duds\right) \\
&\leq \mathbf{E}\left(\lambda^2 n^{\alpha+H_1-1} c(H_2)\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} K\left(n^\alpha B_i^{H_1}\right)K\left(n^\alpha B_j^{H_1}\right)\int_i^{i+1}\int_j^{j+1}|s-u|^{2H_2-2}duds\right)
\end{aligned}$$

and this goes to zero as in the proof showing that the non-diagonal term goes to zero under the renormalization $n^{\alpha+H_1-1}$. Let us now handle the term $T_1^\star$. By using the independence of B^{H_1} and B^{H_2} we can write

$$T_1^\star = \mathbf{E}\left(i_0\lambda_n\int_0^n e^{i_0\lambda_n\eta_s^{(\lambda_n)}} K(n^\alpha B_{[s]}^{H_1}) e(\lambda,n) dB_s^{H_2}\cdot g_N\right).$$

The duality formula is used to obtain

$$\begin{aligned}
T_1^\star &= \mathbf{E}\left(i_0\lambda_n\langle \mathbf{1}_{[0,n]} e^{i_0\lambda_n\eta_\cdot^{(\lambda_n)}} K(n^\alpha B_{[\cdot]}^{H_1}) e(\lambda,n), D^{H_2} g_N\rangle_{\mathcal{H}_{H_2}}\right) \\
&= \mathbf{E}\left(-\lambda_n\langle \mathbf{1}_{[0,n]} e^{i_0\lambda_n\eta_\cdot^{(\lambda_n)}} K(n^\alpha B_{[\cdot]}^{H_1}) e(\lambda,n), g_N\sum_{k=1}^N \beta_k D^{H_2} G_{t_k}\rangle_{\mathcal{H}_{H_2}}\right).
\end{aligned}$$

Recall that the following formula holds (see [Nua06] for further details)

$$\langle\phi,\psi\rangle_{\mathcal{H}_{H_2}} = H_2(2H_2-1)\int_0^T\int_0^T|r-u|^{2H_2-2}\,\phi_r\psi_u dudr$$

for any pair of functions in the Hilbert space $\mathcal{H}_{H_2}$. This formula is used to write $T_1^\star$ as

$$T_1^\star = \mathbf{E}\left(-\lambda_n\sum_{k=1}^N\beta_k H_2(2H_2-1)\int_0^n\int_0^{t_k} e^{i_0\lambda_n\eta_u} K(n^\alpha B_{[u]}^{H_1}) e(\lambda,n) D_v G_{t_k}|u-v|^{2H_2-2}dvdu\right)$$

where the fact that G_t is adapted to the filtration of B^{H_2} is used. It suffices to show that for every fixed $t \geq 0$,

$$\lambda_n \mathbf{E}\left(\int_0^n \int_0^t e^{i_0 \lambda_n \eta_u} K(n^\alpha B_{[u]}^{H_1}) e(\lambda, n) D_v G_t |u-v|^{2H_2-2} dv du\right)$$

converges to zero as $n \to \infty$. Since the derivative of G_t is bounded and using (1.38) we find that the above term is less than

$$\begin{aligned}
& \lambda_n c_1 \mathbf{E}\left(\int_0^n \int_0^t K(n^\alpha B_{[u]}^{H_1}) |u-v|^{2H_2-2} dv du\right) \\
= \quad & \lambda_n c_1 \sum_{i=0}^{n-1} \mathbf{E}\left(K(n^\alpha B_i^{H_1})\right) \int_i^{i+1} \int_0^t |u-v|^{2H_2-2} dv du \\
= \quad & \lambda_n c_1 c_{H_2} \sum_{i=0}^{n-1} \mathbf{E}\left(K(n^\alpha B_i^{H_1})\right) \left(-|i+1-t|^{2H_2} + |i-t|^{2H_2} + |i+1|^{2H_2} - i^{2H_2}\right)
\end{aligned}$$

where c_1 is the constant upper bound of the derivative of G_t and c_{H_2} is a constant depending only on H_2. Since for every fixed $t > 0$ the function

$$\left(-|i+1-t|^{2H_2} + |i-t|^{2H_2} + |i+1|^{2H_2} - i^{2H_2}\right)$$

behaves, modulo a constant, as i^{2H_2-2} and since the order of the expectation of $K(n^\alpha B_i^{H_1})$ is the same as that of $n^{-\alpha} i^{-H_1}$ it is clear that T_1 converges to zero as $n \to \infty$.

Finally, we will show that $T_3^\star$ converges to zero. Note that the term $T_3^\star$ can be expressed as follows

$$T_3^\star \quad = \quad \lambda_n^2 \sum_{i=0}^{n-1} \mathbf{E}\left(K^2(n^\alpha B_i^{H_1}) \int_i^{i+1} e^{i_0 \lambda_n \eta_s^{(\lambda_n)}} \left(\frac{1}{2} - H_2(s-i)^{2H_2-1}\right) ds \cdot e(\lambda, n) g_N\right).$$

At this point, we will again apply the Itô formula for $e^{i\lambda_n \eta_s^{(\lambda_n)}}$. It implies that

$$\begin{aligned}
T_3^\star \quad = \quad & \mathbf{E}\left(\lambda_n^2 \sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right) \int_i^{i+1} \left(\frac{1}{2} - H_2(s-i)^{2H_2-1}\right) ds \cdot e(\lambda, n) g_N\right) \\
& + \mathbf{E}\left(i_0 \lambda_n^3 \sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right) \int_i^{i+1} \left(\frac{1}{2} - H_2(s-i)^{2H_2-1}\right)\right. \\
& \left.\int_0^s e^{i_0 \lambda_n \eta_u^{(\lambda_n)}} K(n^\alpha B_{[u]}^{H_1}) dB_u^{H_2} ds \cdot e(\lambda, n) g_N\right) \\
& + \mathbf{E}\left(\frac{1}{2} \lambda_n^4 \sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right) \int_i^{i+1} \left(\frac{1}{2} - H_2(s-i)^{2H_2-1}\right)\right. \\
& \left.\int_0^s e^{i_0 \lambda_n \eta_u^{(\lambda_n)}} K^2(n^\alpha B_{[u]}^{H_1}) du ds \cdot e(\lambda, n) g_N\right) \\
& - \mathbf{E}\left(\lambda_n^4 \sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right) \int_i^{i+1} \left(\frac{1}{2} - H_2(s-i)^{2H_2-1}\right)\right. \\
& \times H_2(2H_2-1) \int_0^s e^{i_0 \lambda_n \eta_u^{(\lambda_n)}} K(n^\alpha B_{[u]}^{H_1}) \\
& \left.\times \int_0^u K(n^\alpha B_{[v]}^{H_1}) \, |u-v|^{2H_2-2} \, dv du ds \cdot e(\lambda, n) g_N du\right) \\
= \quad & b^{(1)} + b^{(2)} + b^{(3)} + b^{(4)}.
\end{aligned} \tag{1.39}$$

The first summand $b^{(1)}$ vanishes because the integral

$$\int_i^{i+1} \left(\frac{1}{2} - H_2 (s-i)^{2H_2 - 1} \right) ds$$

vanishes. The second summand $b^{(2)}$ goes to zero as $n \to \infty$ using exactly the same argument as for the convergence of $T_1^\star$. Concerning the third summand, $b^{(3)}$, using (1.38) and the fact that $|g_N| = 1$, we get

$$\begin{aligned}
b^{(3)} &\leq \mathbf{E}\left(\lambda_n^4 \sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right) \int_i^{i+1} \underbrace{\left| \frac{1}{2} - H_2(s-i)^{2H_2-1} \right|}_{\leq 1} \int_0^s K^2(n^\alpha B_{[u]}^{H_1}) du ds \right) \\
&\leq \mathbf{E}\left(\frac{1}{2}\lambda_n^4 \sum_{i=0}^{n-1} K^2\left(n^\alpha B_i^{H_1}\right) \int_i^{i+1} \left(\sum_{j=0}^{i-1} K^2(n^\alpha B_j^{H_1}) + K^2(n^\alpha B_i^{H_1}) \underbrace{(s-i)}_{\leq 1} \right) ds \right) \\
&\leq \mathbf{E}\left(\lambda_n^4 \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} K^2\left(n^\alpha B_i^{H_1}\right) K^2\left(n^\alpha B_j^{H_1}\right) \right) + \mathbf{E}\left(\lambda_n^4 \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} K^4\left(n^\alpha B_i^{H_1}\right) \right).
\end{aligned}$$

The second term goes to zero because $\mathbf{E}\left(\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} K^4\left(n^\alpha B_i^{H_1}\right)\right)$ behaves as $n^{-\alpha - H_1 + 1}$ and the first term goes to zero because the non-diagonal term is dominated by the diagonal term. Analogously to the convergence of $T_2^\star$, the last summand in (1.39) converges to zero. This completes the proof. ■

Deuxième partie

Error bounds in CLTs via Stein's Method and Malliavin calculus

Chapitre 2

Berry-Esséen Bounds for Long Memory Moving Averages via Stein's Method and Malliavin Calculus

Solesne BOURGUIN * and Ciprian A. TUDOR † ‡
Université Paris 1 and *Université Lille 1*

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Abstract

Using the Stein method on Wiener chaos introduced in [NP09c] we prove Berry-Esséen bounds for long memory moving averages.

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2.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t\geq 0}$ be a Brownian motion on this space. Let F be a random variable defined on Ω which is differentiable in the sense of Malliavin calculus. Then, using Stein's method on Wiener chaos, introduced by Nourdin and Peccati in [NP09c] (see also [NP09b] and [NP10]), it is possible to measure the distance between the law of F and the standard normal law $N(0,1)$. This distance can be defined in several ways (the Kolmogorov distance, the Wasserstein distance, the total variation distance or

∗. SAMM, Université de Paris 1 Panthéon-Sorbonne, 90, rue de Tolbiac, 75634, Paris, France. Email : solesne.bourguin@univ-paris1.fr

†. Laboratoire Paul Painlevé, Université de Lille 1, F-59655 Villeneuve d'Ascq, France. Email : tudor@math.univ-lille1.fr

‡. Associate member of the team Samm, Université de Paris 1 Panthéon-Sorbonne.

the Fortet-Mourier distance). More precisely we have, if $\mathcal{L}(F)$ denotes the law of F and F is centered,

$$d(\mathcal{L}(F), N(0,1)) \leq c\sqrt{E\left(1 - \langle DF, D(-L)^{-1}F\rangle_{L^2([0,1])}\right)^2}. \tag{2.1}$$

Here, D denotes the Malliavin derivative with respect to W while L is the generator of the Ornstein-Uhlenbeck semigroup. We will explain in the next section how these operators are defined. The constant c is equal to 1 in the case of the Kolmogorov and of the Wasserstein distance, c=2 for the total variation distance and $c = 4$ in the case of the Fortet-Mourier distance.

These results have already been used to prove error bounds in various central limit theorems. In [NP09c] the authors prove Berry-Esséen bounds in the central limit theorem for the subordinated functionals of the fractional Brownian motion and [NP09b] focuses on central limit theorems for Toeplitz quadratic functionals of continuous-time stationary processes. In [NPR10] the authors extended the Stein's method to multidimensional settings. See also [AMV10].

In this paper we will consider long memory moving averages defined by

$$X_n = \sum_{i\geq 1} a_i \varepsilon_{n-i}, n \in \mathbb{Z}$$

where the innovations ε_i are centered i.i.d. random variables having at least finite second moments and the moving averages a_i are of the form $a_i = i^{-\beta}L(i)$ with $\beta \in (\frac{1}{2}, 1)$ and L slowly varying towards infinity. The covariance function $\rho(m) = \mathbf{E}\left(X_0 X_m\right)$ behaves as $c_\beta m^{-2\beta+1}$ when $m \to \infty$ and consequently is not summable since $\beta < 1$. Therefore X_n is usually called long-memory or "long-range dependence" moving average. Let K be a deterministic function which has Hermite rank q and satisfies $\mathbf{E}(K^2(X_n)) < \infty$ and define

$$S_N = \sum_{n=1}^{N} \left[K(X_n) - \mathbf{E}\left(K(X_n)\right)\right].$$

Then it has been proven in [HH97] (see also [Wu06]) that, with $c_1(\beta, q), c_2(\beta, q)$ being positive constants depending only on q and β : a) If $q > \frac{1}{2\beta-1}$, then the sequence $c_1(\beta, q)\frac{1}{\sqrt{N}}S_N$ converges in law to a standard normal random variable and b) If $q < \frac{1}{2\beta-1}$, then the sequence $c_2(\beta, q)N^{\beta q - \frac{q}{2} - 1}S_N$ converges in law to a Hermite random variable of order q. This Hermite random variable, which will be defined in the next section, is actually an iterated integral of a deterministic function with q variables with respect to a Wiener process. This theorem is a variant of the non-central limit theorem from [DM79] and [Taq79]. In order to apply the techniques based on the Malliavin calculus and multiple Wiener-Itô integrals, we will restrict our focus to the following situation : the innovations ε_i are chosen to be the increments of a Brownian motion W on the real line while the function K is a Hermite polynomial of order q. In this case the random variable X_n is a Wiener integral with respect to W, and $H_q(X_n)$ can be expressed as a multiple Wiener-Itô stochastic integral of order q with respect to W. When $q > \frac{1}{2\beta-1}$ we will apply formula (2.1) in order to obtain the rate of convergence of S_N. When $q < \frac{1}{2\beta-1}$ the limit of S_N (after normalization) is not Gaussian and so we will use a different argument based on a result in [DM89] that has already been exploited in [BN08].

The paper is organized as follows. Section 2 deals with notation and preliminaries, such as the definition of a moving average process and a Wiener process on $\mathbb{R}$, but also gives a brief introduction to the tools of Malliavin calculus. In section 3, we will prove the Berry-Esséen bounds for the central and non central limit theorems for long-memory moving averages. Section 4 shows an application of our results to the Hsu-Robbins and Spitzer theorems for moving averages.

2.2 Notation and Preliminaries

In this section, we will give the main properties of infinite moving average processes and a proper definition of a Brownian motion on $\mathbb{R}$. We will relate one to the other to prove that the processes that we will consider in the sequel are well defined. To conclude the preliminaries, we will finally focus on the sequences and results, such as central and non-central limit theorems that interest us in this paper.

2.2.1 The Infinite Moving Average Process

Before introducing the infinite moving average process, we will need the proper definition of a white noise on $\mathbb{Z}$.

Definition 1. *The process $\{Z_t\}_{t\in\mathbb{Z}}$ is said to be a white noise with zero mean and variance σ^2, written*

$$\{Z_t\} \sim \mathcal{WN}(0,\sigma^2),$$

if and only if $\{Z_t\}$ has zero mean and covariance function $\gamma(h) = \mathbf{E}(Z_{t+h}Z_t)$, $h \in \mathbb{N}$, defined by

$$\gamma(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0. \end{cases}$$

Now we can define the infinite moving average process.

Definition 2. *If $\{Z_t\} \sim \mathcal{WN}(0,\sigma^2)$ then we say that $\{X_t\}$ is a moving average $(MA(\infty))$ of $\{Z_t\}$ if there exists a sequence $\{\psi_j\}$ with $\sum_{j=0}^{\infty}|\psi_j| < \infty$ such that*

$$X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j}, \quad t = 0, \pm 1, \pm 2, ... \tag{2.2}$$

We have the following proposition on infinite moving averages (see [BD91] p. 91).

Proposition 5. *The $(MA(\infty))$ process defined by (2.2) is stationary with mean zero and covariance function*

$$\gamma(k) = \sigma^2 \sum_{j=0}^{\infty}\psi_j\psi_{j+|k|}. \tag{2.3}$$

For further details on moving averages, see [BD91] for a complete survey of this topic.

2.2.2 The Brownian Motion on $\mathbb{R}$

Here, we will give a proper definition of a two-sided Brownian motion on $\mathbb{R}$. We will then connect this definition to the underlying Hilbert space.

Definition 3. *A two sided Brownian motion $\{W_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}$ is a continous centered Gaussian process with covariance function*

$$R(t,s) = \frac{1}{2}\left(|s| + |t| - |t-s|\right), \quad s,t \in \mathbb{R}. \tag{2.4}$$

Let $\mathcal{H} = L^2(\mathbb{R})$ be the underlying Hilbert space of this particular process. We have

$$R(t,s) = \begin{cases} \left\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \right\rangle_{\mathcal{H}} = s \wedge t \textit{ if } \ s,t \geq 0 \\ \left\langle \mathbf{1}_{[s,0]}, \mathbf{1}_{[0,t]} \right\rangle_{\mathcal{H}} = 0 \textit{ if } \ s \leq 0 \textit{ and } \ t \geq 0 \\ \left\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[t,0]} \right\rangle_{\mathcal{H}} = 0 \textit{ if } \ s \geq 0 \textit{ and } \ t \leq 0 \\ \left\langle \mathbf{1}_{[s,0]}, \mathbf{1}_{[t,0]} \right\rangle_{\mathcal{H}} = -(s \vee t) = |s| \wedge |t| \ \textit{ if } \ s,t \leq 0. \end{cases} \tag{2.5}$$

We could also define the two-sided Brownian motion by considering two independent standard Brownian motions on $\mathbb{R}^+$, $\left\{W_t^{(1)}\right\}$ and $\left\{W_t^{(2)}\right\}$ and by setting

$$W_t = \begin{cases} W_t^{(1)} \textit{ if } \ t \geq 0 \\ W_{-t}^{(2)} \textit{ if } \ t \leq 0. \end{cases} \tag{2.6}$$

$\{W_t\}$ has the same law as the one induced by the first definition.

If we define the process $\{I_t\}_{t\in\mathbb{Z}}$ as the increment of the two-sided Brownian motion between t and $t+1$, $t \in \mathbb{Z}$, we have $I_t = W_{t+1} - W_t$. Then the process $\{I_t\}_{t\in\mathbb{Z}}$ is a white noise on $\mathbb{Z}$ with mean 0 and variance 1. Indeed, It is clear that $\{I_t\}$ is a centered Gaussian process. We only need to verify its covariance function. We have, for every $h \in \mathbb{Z}$,

$$\mathbf{E}\left(I_{t+h} I_t\right) = \mathbf{E}\left((W_{t+h+1} - W_{t+h})(W_{t+1} - W_t)\right) = \begin{cases} 1 \text{ if } \ h = 0 \\ 0 \text{ if } \ h \neq 0 \end{cases}$$

2.2.3 Limit Theorems for Functionals of i.i.d Gaussian Processes

Here, we will focus on the following type of sequences

$$S_N = \sum_{n=1}^{N} \left[K(X_n) - \mathbf{E}\left(K(X_n)\right)\right] \tag{2.7}$$

where

$$X_n = \sum_{i=1}^{\infty} \alpha_i \left(W_{n-i} - W_{n-i-1}\right), \tag{2.8}$$

with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{\infty} \alpha_i^2 = 1$. Note that $\{X_n\}$ is an infinite moving average of the white noise $\{I_t\} = \{W_{t+1} - W_t\}$. Thus its covariance function is given by

$$\rho(m) := \sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|}. \tag{2.9}$$

For those sequences, central and non-central limit theorems have been proven. Here are the main results we will be focusing on.

Theorem 5. *Suppose that the α_i are regularly varying with exponent $-\beta$, $\beta \in (1/2, 1)$ (i.e. $\alpha_i = |i|^{-\beta} L(i)$ and that $L(i)$ is slowly varying at ∞). Suppose that K has Hermite rank k and satisfies $\mathbf{E}(K^2(X_n)) < \infty$. Then*

i. If $k < (2\beta - 1)^{-1}$, then

$$h_{k,\beta}^{-1} N^{\beta k - \frac{k}{2} - 1} S_N \underset{N \to +\infty}{\longrightarrow} Z^{(k)} \tag{2.10}$$

where $Z^{(k)}$ is a Hermite random variable of order k defined by (2.14) and $h_{k,\beta}$ is a positive constant depending on k and β (which will be defined later by (2.33)).

ii. If $k > (2\beta - 1)^{-1}$, then

$$\frac{1}{\sigma_{k,\beta}\sqrt{N}} S_N \underset{N \to +\infty}{\longrightarrow} \mathcal{N}(0, 1) \tag{2.11}$$

with $\sigma_{k,\beta}$ defined by (2.24).

We will compute the Berry-Esséen bounds for these central limit (CLT) and non-central limit (NCLT) theorems using Stein's Method and Malliavin Calculus. In the next paragraph, we will give the basic elements on these topics.

2.2.4 Multiple Wiener-Itô Integrals and Malliavin Derivatives

Here we describe the elements from stochastic analysis that we will need in the paper. Consider $\mathcal{H}$ a real separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$. Denote I_n the multiple stochastic integral with respect to B (see [Nua06]). This I_n is actually an isometry between the Hilbert space $\mathcal{H}^{\odot n}$(symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(B(\varphi))$ where $\varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1$ and H_n is the Hermite polynomial of degree $n \geq 1$

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as : for m, n positive integers,

$$\begin{aligned} \mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{2.12}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by B can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n \geq 0} I_n(f_n) \tag{2.13}$$

where $f_n \in \mathcal{H}^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n\geq 0} nI_n(f_n)$$

if F is given by (5.3)and it is such that $\sum_{n=1}^{\infty} n^2 n\|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty$.

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|((I-L)F)^{\frac{\alpha}{2}}\|_{L^p(\Omega)}$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in \mathcal{H}$)

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(\mathcal{H})$.

In this paper we will use the Malliavin calculus with respect to the Brownian motion on $\mathbb{R}$ as introduced above. Note that the Brownian motion on the real line is an isonormal process and its underlying Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$.

We will now introduce the Hermite random variable, which is the limit in Theorem 1, point i. The Hermite random variable of order q is given by

$$Z^{(q)} = d(q,\beta)I_q(g(\cdot)) \tag{2.14}$$

where

$$g(y_1, .., y_q) = \int_{y_1\vee\ldots\vee y_q}^{1} du(u-y_1)_+^{-\beta}\ldots(u-y_q)_+^{-\beta}. \tag{2.15}$$

The constant $d(q,\beta)$ is a normalizing constant which ensures that $\mathbf{E}(Z^{(q)})^2 = 1$. This constant is explicitly computed below.

$$\begin{aligned}
\mathbf{E}(Z^{(q)})^2 &= q!d(q,\beta)^2 \int_{[0,1]^2} dudv \left(\int_{\mathbb{R}} (u-y)_+^{-\beta}(v-y)_+^{-\beta} dy\right)^q \\
&= q!d(q,\beta)^2 \boldsymbol{\beta}(2\beta-1, 1-\beta)^q \int_{[0,1]^2} dudv|u-v|^{-2q\beta+q} \\
&= q!d(q,\beta)^2 \boldsymbol{\beta}(2\beta-1, 1-\beta)^q \frac{2}{(-2\beta q+q+1)(-2\beta q+q+2)}
\end{aligned}$$

where we used

$$\int_{\mathbb{R}} (u-y)_+^{-\beta}(v-y)_+^{-\beta} dy = \boldsymbol{\beta}(2\beta-1, 1-\beta)|u-v|^{-2\beta+1} = c_\beta|u-v|^{-2\beta+1}$$

and we denoted $c_\beta := \boldsymbol{\beta}(2\beta-1, 1-\beta)$, $\boldsymbol{\beta}$ being the beta function defined by

$$\boldsymbol{\beta}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt.$$

Therefore

$$d(q,\beta)^2 = \frac{(-2\beta q+q+1)(-2\beta q+q+2)}{2q!c_\beta^q}. \tag{2.16}$$

2.2.5 Stein's Method on a Fixed Wiener Chaos

Let $F = I_q(h), h \in \mathcal{H}^{\odot q}$ be an element on the Wiener chaos of order q. Recall that for any fixed $z \in \mathbb{R}$, the Stein equation is given by

$$\mathbf{1}_{(\infty,z]}(x) - \Phi(z) = f'(x) - xf(x). \tag{2.17}$$

It is well known that (2.17) admits a solution f_z bounded by $\sqrt{2\pi}/4$ and such that $\|f_z'\|_\infty \leq 1$. By taking $x = F$ in (2.17) and by taking the expectation, we get

$$\mathbf{P}(F \leq z) - \mathbf{P}(N \leq z) = \mathbf{E}\left(f_z'(F) - Ff_z(F)\right) \tag{2.18}$$

where N is a standard normal random variable ($N \hookrightarrow \mathcal{N}(0,1)$). By writing $F = LL^{-1}F = -\delta DL^{-1}F$ and by integrating by part, we find

$$\begin{aligned}
\mathbf{E}\left(Ff_z(F)\right) &= \mathbf{E}\left(-\delta DL^{-1}Ff_z(F)\right) = \mathbf{E}\left(\left\langle -DL^{-1}F, D(f_z(F))\right\rangle_{\mathcal{H}}\right) \\
&= \mathbf{E}\left(\left\langle -DL^{-1}F, f_z'(F)DF\right\rangle_{\mathcal{H}}\right) = \mathbf{E}\left(f_z'(F)\left\langle DF, -DL^{-1}F\right\rangle_{\mathcal{H}}\right).
\end{aligned}$$

Thus, by replacing in (2.18), we obtain

$$\mathbf{E}\left(f_z'(F) - Ff_z(F)\right) = \mathbf{E}\left(f_z'(F)\left(1 - \left\langle DF, -DL^{-1}F\right\rangle_{\mathcal{H}}\right)\right)$$

and

$$\mathbf{P}(F \leq z) - \mathbf{P}(N \leq z) = \mathbf{E}\left(f_z'(F)\left(1 - \left\langle DF, -DL^{-1}F\right\rangle_{\mathcal{H}}\right)\right). \tag{2.19}$$

On the other hand, the Kolmogorov distance is defined by

$$d_{\mathrm{Kol}}(X,Y) = \sup_{z\in\mathbb{R}} \left|\mathbf{P}(X \leq z) - \mathbf{P}(Y \leq z)\right|. \tag{2.20}$$

Therefore we have

$$d_{\mathrm{Kol}}(F,N) = \sup_{z\in\mathbb{R}} \left|\mathbf{E}\left(f_z'(F)\left(1 - \left\langle DF, -DL^{-1}F\right\rangle_{\mathcal{H}}\right)\right)\right|.$$

By applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
d_{\mathrm{Kol}}(F,N) &\leq \underbrace{\left[\mathbf{E}\left((f_z'(F))^2\right)\right]^{\frac{1}{2}}}_{\leq 1}\left[\mathbf{E}\left(\left(1 - \left\langle DF, -DL^{-1}F\right\rangle_{\mathcal{H}}\right)^2\right)\right]^{\frac{1}{2}} \\
&\leq \sqrt{\mathbf{E}\left((1 - \langle DF, -DL^{-1}F\rangle_{\mathcal{H}})^2\right)}.
\end{aligned} \tag{2.21}$$

Recall that $F = I_q(h)$ and so in that case the equality

$$\left\langle DF, -DL^{-1}F\right\rangle_{\mathcal{H}} = q^{-1}\left\|DF\right\|_{\mathcal{H}}^2$$

holds. Thus, we can rewrite (2.21) as

$$d_{\mathrm{Kol}}(F,N) \leq c\sqrt{\mathbf{E}\left(\left(1 - q^{-1}\left\|DF\right\|_{\mathcal{H}}^2\right)^2\right)} \tag{2.22}$$

with $c = 1$. As we mentioned in the introduction, the above inequality remains true for other distances (Wasserstein, total variation or Fortet-Mourier). The constant c is equal to 1 in the case of the Kolmogorov and of the Wasserstein distance, c=2 for the total variation distance and $c = 4$ in the case of the Fortet-Mourier distance.

2.3 Berry-Esséen Bounds in the Central and Non-Central Limit Theorems

As previously mentionned in the introduction, we will focus on the case where $K = H_q$, H_q being the Hermite polynomial of order q. In this case, we will be able to give a more appropriate representation of S_N in terms of multiple stochastic integrals. We will also assume $\alpha_i \sim i^{-\beta}$ for large i.

2.3.1 Representation of S_N as an Element of the q^{th}-Chaos

Note that X_n can also be written as

$$
\begin{aligned}
X_n &= \sum_{i=1}^{\infty} \alpha_i \left(W_{n-i} - W_{n-i-1}\right) = \sum_{i=1}^{\infty} \alpha_i I_1\left(\mathbf{1}_{[n-i-1,n-i]}\right) \\
&= I_1\left(\underbrace{\sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{[n-i-1,n-i]}}_{f_n}\right) = I_1\left(f_n\right).
\end{aligned}
\tag{2.23}
$$

As $K = H_q$, we have

$$
S_N = \sum_{n=1}^{N} \left[H_q(X_n) - \mathbf{E}\left(H_q(X_n)\right)\right] = \sum_{n=1}^{N} \left[H_q(I_1(f_n)) - \mathbf{E}\left(H_q(I_1(f_n))\right)\right]
$$

We know that, if $\|f\|_{\mathcal{H}} = 1$, we have $H_q(I_1(f)) = \frac{1}{q!} I_q(f^{\otimes q})$. Furthermore, we have

$$
\begin{aligned}
\|f_n\|_{\mathcal{H}}^2 &= \langle f_n, f_n \rangle_{\mathcal{H}} = \left\langle \sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{[n-i-1,n-i]}, \sum_{r=1}^{\infty} \alpha_r \mathbf{1}_{[n-r-1,n-r]} \right\rangle_{\mathcal{H}} \\
&= \sum_{i,r=1}^{\infty} \alpha_i \alpha_r \left\langle \mathbf{1}_{[n-i-1,n-i]}, \mathbf{1}_{[n-r-1,n-r]} \right\rangle_{\mathcal{H}}.
\end{aligned}
$$

It is easily verified that if $i > r \Leftrightarrow n - i \leq n - r - 1$ or $i < r \Leftrightarrow n - r \leq n - i - 1$, we have $[n-i-1, n-i] \cap [n-r-1, n-r] = \emptyset$ and thus $\left\langle \mathbf{1}_{[n-i-1,n-i]}, \mathbf{1}_{[n-r-1,n-r]} \right\rangle_{\mathcal{H}} = 0$. It follows that

$$
\|f_n\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \alpha_i^2 \left\| \mathbf{1}_{[n-i-1,n-i]} \right\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \alpha_i^2 = 1.
$$

Thanks to this result, S_N can be represented as

$$
\begin{aligned}
S_N &= \sum_{n=1}^{N} \left[H_q(I_1(f_n)) - \mathbf{E}\left(H_q(I_1(f_n))\right)\right] = \frac{1}{q!} \sum_{n=1}^{N} \left[I_q(f_n^{\otimes q}) - \mathbf{E}\left(I_q(f_n^{\otimes q})\right)\right] \\
&= \frac{1}{q!} \sum_{n=1}^{N} I_q(f_n^{\otimes q}) = \frac{1}{q!} I_q\Big(\sum_{n=1}^{N} f_n^{\otimes q}\Big).
\end{aligned}
$$

2.3.2 Berry-Esséen Bounds for the Central Limit Theorem

We will first focus on the case where $q > (2\beta-1)^{-1}$, i.e. the central limit theorem. Let $Z_N = \frac{1}{\sigma\sqrt{N}} S_N$ where $\sigma_{q,\beta}$ is given by

$$\sigma := \sigma_{q,\beta}^2 = q! \sum_{m=-\infty}^{+\infty} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q = q! \sum_{m=-\infty}^{+\infty} \rho^q(m). \tag{2.24}$$

The following result gives the Berry-Esséen bounds for the central limit part of theorem 5.

Theorem 6. *Under the condition $q > (2\beta-1)^{-1}$, Z_N converges in law towards $Z \sim \mathcal{N}(0,1)$. Moreover, there exists a constant C_β, depending uniquely on β, such that, for any $N \geq 1$,*

$$\underset{z\in\mathbb{R}}{sup} |\mathbf{P}(Z_N \leq z) - \mathbf{P}(Z \leq z)| \leq C_\beta \begin{cases} N^{\frac{q}{2}+\frac{1}{2}-q\beta} & \text{if } \beta \in \left(\frac{1}{2}, \frac{q}{2q-2}\right] \\ N^{\frac{1}{2}-\beta} & \text{if } \beta \in \left[\frac{q}{2q-2}, 1\right) \end{cases}$$

Remark 2. *The same result, modulo the change of the constant, holds for other distances between the laws of random variables (e.g. total variations distance, Wasserstein etc. See the Introduction, see also [NP09c])*

Proof : Theorem 5 states that $Z_N \underset{N\to+\infty}{\longrightarrow} \mathcal{N}(0,1)$. Thanks to (2.21) and (2.22), we will evaluate the quantity

$$\mathbf{E}\left(\left(1 - q^{-1} \|DZ_N\|_{\mathcal{H}}^2\right)^2 \right).$$

We will start by computing $\|DZ_N\|_{\mathcal{H}}^2$. We have the following lemma.

Lemma 5. *The following result on $\|DZ_N\|_{\mathcal{H}}$ holds.*

$$\frac{1}{q} \|DZ_N\|_{\mathcal{H}}^2 - 1 \quad = \quad \sum_{r=0}^{q-1} A_r(N) - 1$$

where

$$A_r(N) = \frac{qr!}{\sigma^2 N} \begin{pmatrix} q-1 \\ r \end{pmatrix}^2 \sum_{k,l=1}^{N} I_{2q-2-2r}\left(f_k^{\otimes q-1-r} \widetilde{\otimes} f_l^{\otimes q-1-r} \right) \langle f_k, f_l \rangle_{\mathcal{H}}^{r+1}. \tag{2.25}$$

Proof : We have

$$DZ_N \quad = \quad D\left(\frac{1}{\sigma\sqrt{N}} \sum_{n=1}^{N} I_q\left(f_n^{\otimes q}\right) \right) = \frac{q}{\sigma\sqrt{N}} \sum_{n=1}^{N} I_{q-1}\left(f_n^{\otimes q-1}\right) f_n$$

and

$$\|DZ_N\|_{\mathcal{H}}^2 \quad = \quad \frac{q^2}{\sigma^2 N} \sum_{k,l=1}^{N} I_{q-1}\left(f_k^{\otimes q-1}\right) I_{q-1}\left(f_l^{\otimes q-1}\right) \langle f_k, f_l \rangle_{\mathcal{H}}. \tag{2.26}$$

The multiplication formula between multiple stochastic integrals gives us that

$$I_{q-1}\left(f_k^{\otimes q-1}\right) I_{q-1}\left(f_l^{\otimes q-1}\right) \quad = \quad \sum_{r=0}^{q-1} r! \begin{pmatrix} q-1 \\ r \end{pmatrix}^2 I_{2q-2-2r}\left(f_k^{\otimes q-1-r} \widetilde{\otimes} f_l^{\otimes q-1-r} \right) \langle f_k, f_l \rangle_{\mathcal{H}}^{r}.$$

By replacing in (2.26), we obtain

$$\|DZ_N\|_{\mathcal{H}}^2 = \frac{q^2}{\sigma^2 N}\sum_{r=0}^{q-1} r! \begin{pmatrix} q-1 \\ r \end{pmatrix}^2 \sum_{k,l=1}^{N} I_{2q-2-2r}\left(f_k^{\otimes q-1-r}\tilde{\otimes} f_l^{\otimes q-1-r}\right)\langle f_k, f_l\rangle_{\mathcal{H}}^{r+1}$$

and the conclusion follows easily. ■

By using Lemma 5 and the fact that $\mathbf{E}(I_m I_n) = 0$ if $m \neq n$, we can now evaluate $\mathbf{E}\left(\left(1 - q^{-1}\|DZ_N\|_{\mathcal{H}}^2\right)^2\right)$. We have

$$\mathbf{E}\left(\left(1 - q^{-1}\|DZ_N\|_{\mathcal{H}}^2\right)^2\right) = \sum_{r=0}^{q-2}\mathbf{E}\left(A_r^2(N)\right) + \mathbf{E}(A_{q-1}(N)-1)^2. \tag{2.27}$$

We need to evaluate the behaviour of those two terms as $N \to \infty$, but first, recall that the α_i are of the form $\alpha_i = i^{-\beta}$ with $\beta \in (1/2, 1)$. We will use the notation $a_n \sim b_n$ meaning that a_n and b_n have the same limit as $n \to \infty$ and $a_n \leqslant b_n$ meaning that $\sup_{n\geq 1}|a_n|/|b_n| < \infty$. Below is a useful lemma we will use throughout the paper.

Lemma 6. *1. We have*

$$\rho(n) \sim c_\beta n^{-2\beta+1}$$

with $c_\beta = \int_0^\infty y^{-\beta}(y+1)^{-\beta}dy = \boldsymbol{\beta}(2\beta-1, 1-\beta)$. The constant c_β is the same as the one in the definition of the Hermite random variable (see (2.16)).

2. For any $\alpha \in \mathbb{R}$, we have

$$\sum_{k=1}^{n-1} k^\alpha \leqslant 1 + n^{\alpha+1}.$$

3. If $\alpha \in (-\infty, -1)$, we have

$$\sum_{k=n}^{\infty} k^\alpha \leqslant n^{\alpha+1}.$$

Proof : Points 2. and 3. follow from [NP09c], Lemma 4.3. We will only prove the first point of the lemma (as the other points have been proven in [NP09c]). We know that $\rho(n) = \sum_{i=1}^\infty i^{-\beta}(i+|n|)^{-\beta}$ behaves as $\int_0^\infty x^{-\beta}(x+|n|)^{-\beta}dx$ and the following holds

$$\int_0^\infty x^{-\beta}(x+|n|)^{-\beta}dx = \int_0^\infty x^{-\beta}|n|^{-\beta}\left(\frac{x}{|n|}+1\right)^{-\beta}dx = |n|^{-2\beta+1}\underbrace{\int_0^\infty y^{-\beta}(y+1)^{-\beta}dy}_{c_\beta}.$$

Thus,

$$\rho(n) \sim \sum_{i=1}^\infty i^{-\beta}(i+|n|)^{-\beta} \sim c_\beta n^{-2\beta+1}.$$

■

We will start the evaluation of (2.27) with the term $\mathbf{E}(A_{q-1}(N)-1)^2$. Note that we have $\mathbf{E}(A_{q-1}(N)-1)^2 = (A_{q-1}(N)-1)^2$ because $A_{q-1}(N)-1$ is deterministic. We can write

$$A_{q-1}(N) - 1 = \frac{q!}{\sigma^2 N}\sum_{k,l=1}^N \langle f_k, f_l\rangle_{\mathcal{H}}^q - 1.$$

Note that we have

$$\langle f_k, f_l\rangle_{\mathcal{H}} = \sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} = \rho(l-k).$$

Hence

$$\begin{aligned}
& A_{q-1}(N) - 1 \\
&= \frac{q!}{\sigma^2 N} \sum_{k,l=1}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q - 1 \\
&= \frac{1}{\sigma^2 N} \left(q! \sum_{k,l=1}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q - N\sigma^2 \right) \\
&= \frac{1}{\sigma^2 N} \left(q! \sum_{k,l=1}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q - Nq! \sum_{m=-\infty}^{+\infty} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q \right). \qquad (2.28)
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{k,l=1}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q &= \sum_{k\le l}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q + \sum_{k>l}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q \\
&= \sum_{k=1}^{N} \sum_{l=k}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q + \sum_{l=1}^{N} \sum_{k=l+1}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q .
\end{aligned}$$

Let $m = l - k$. We obtain

$$\begin{aligned}
\sum_{k,l=1}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|l-k|} \right)^q &= \sum_{k=1}^{N} \sum_{m=0}^{N-k} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q + \sum_{l=1}^{N} \sum_{m=-N+l}^{-1} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q \\
&= \sum_{m=0}^{N-1} \sum_{k=1}^{N-m} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q + \sum_{m=-(N-1)}^{-1} \sum_{l=1}^{N+m} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q \\
&= \sum_{m=0}^{N-1} (N-m) \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q \\
&\quad + \sum_{m=-(N-1)}^{-1} (N+m) \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q \\
&= N \sum_{m=-(N-1)}^{N-1} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q - 2 \sum_{m=0}^{N-1} m \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|m|} \right)^q .
\end{aligned}$$

By replacing in (2.28), we get

$$\begin{aligned} A_{q-1}(N)-1 &= \frac{q!}{\sigma^2 N}\left(N\sum_{m=-(N-1)}^{N-1}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q - N\sum_{m=-\infty}^{+\infty}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q \right. \\ &\quad \left. -2\sum_{m=0}^{N-1} m\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right) \\ &= \frac{q!}{\sigma^2 N}\left(-N\sum_{m=-\infty}^{-N}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q - N\sum_{m=N}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q \right. \\ &\quad \left. -2\sum_{m=0}^{N-1} m\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right) \\ &= \frac{q!}{\sigma^2 N}\left(-2N\sum_{m=N}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q - 2\sum_{m=0}^{N-1} m\left(\sum_{i=1}^{\infty}\alpha_i\alpha_{i+|m|}\right)^q\right). \end{aligned}$$

By noticing that the condition $q > (2\beta-1)^{-1}$ is equivalent to $-q(2\beta-1) < -1$, we can apply Lemma 6 to get

$$\begin{aligned} A_{q-1}(N)-1 &\leqslant \sum_{m=N}^{\infty} m^{-q(2\beta-1)} + N^{-1}\sum_{m=0}^{N-1} m^{-q(2\beta-1)+1} \\ &\leqslant N^{-q(2\beta-1)+1} + N^{-1}(1+N^{-q(2\beta-1)+2}) \end{aligned}$$

and finally

$$A_{q-1}(N)-1 \leqslant N^{-1} + N^{q-2q\beta+1}.$$

Thus, we obtain a bound on $(A_{q-1}(N)-1)^2 = \mathbf{E}(A_{q-1}(N)-1)^2$,

$$\mathbf{E}(A_{q-1}(N)-1)^2 \leqslant N^{-2} + N^{q-2q\beta} + N^{2q-4q\beta+2}. \tag{2.29}$$

Let us now treat the second term of (2.27), i.e. $\sum_{r=0}^{q-2}\mathbf{E}\left(A_r^2(N)\right)$. Here we can assume that $r \leq q-2$ is fixed. We have

$$\begin{aligned} \mathbf{E}\left(A_r^2(N)\right) &= \mathbf{E}\left(\frac{q^2 r!^2}{\sigma^4 N^2}\begin{pmatrix} q-1 \\ r\end{pmatrix}^4 \sum_{i,j,k,l=1}^{N} \langle f_k, f_l\rangle_{\mathcal{H}}^{r+1}\langle f_i, f_j\rangle_{\mathcal{H}}^{r+1} \right. \\ &\quad \left. \times I_{2q-2-2r}\left(f_k^{\otimes q-1-r}\widetilde{\otimes} f_l^{\otimes q-1-r}\right) I_{2q-2-2r}\left(f_i^{\otimes q-1-r}\widetilde{\otimes} f_j^{\otimes q-1-r}\right)\right) \\ &= c(r,q)N^{-2}\sum_{i,j,k,l=1}^{N}\langle f_k, f_l\rangle_{\mathcal{H}}^{r+1}\langle f_i, f_j\rangle_{\mathcal{H}}^{r+1} \\ &\quad \times\left\langle f_k^{\otimes q-1-r}\widetilde{\otimes} f_l^{\otimes q-1-r}, f_i^{\otimes q-1-r}\widetilde{\otimes} f_j^{\otimes q-1-r}\right\rangle_{\mathcal{H}^{\otimes 2q-2r-2}} \\ &= \sum_{\substack{\alpha,\nu\geq 0 \\ \alpha+\nu=q-r-1}}\ \sum_{\substack{\gamma,\delta\geq 0 \\ \gamma+\delta=q-r-1}} c(r,q,\alpha,\nu,\gamma,\delta)B_{r,\alpha,\nu,\gamma,\delta}(N) \end{aligned}$$

where

$$\begin{aligned} B_{r,\alpha,\nu,\gamma,\delta}(N) &= N^{-2}\sum_{i,j,k,l=1}^{N}\langle f_k, f_l\rangle_{\mathcal{H}}^{r+1}\langle f_i, f_j\rangle_{\mathcal{H}}^{r+1}\langle f_k, f_i\rangle_{\mathcal{H}}^{\alpha}\langle f_k, f_j\rangle_{\mathcal{H}}^{\nu}\langle f_l, f_i\rangle_{\mathcal{H}}^{\gamma}\langle f_l, f_j\rangle_{\mathcal{H}}^{\delta} \\ &= N^{-2}\sum_{i,j,k,l=1}^{N}\rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}. \end{aligned}$$

When α,ν,γ and δ are fixed, we can decompose the sum $\sum_{i,j,k,l=1}^{N}$ which appears in $B_{r,\alpha,\nu,\gamma,\delta}(N)$ just above, as follows :

$$\sum_{i=j=k=l} + \left(\sum_{\substack{i=j=k \\ l\neq i}} + \sum_{\substack{i=j=l \\ k\neq i}} + \sum_{\substack{i=l=k \\ j\neq i}} + \sum_{\substack{j=k=l \\ i\neq j}} \right) + \left(\sum_{\substack{i=j,k=l \\ k\neq i}} + \sum_{\substack{i=k,j=l \\ j\neq i}} + \sum_{\substack{i=l,j=k \\ j\neq i}} \right)$$
$$+ \left(\sum_{\substack{i=j,k\neq i \\ k\neq l,l\neq i}} + \sum_{\substack{i=k,j\neq i \\ j\neq l,k\neq l}} + \sum_{\substack{i=l,k\neq i \\ k\neq j,j\neq i}} + \sum_{\substack{j=k,k\neq i \\ k\neq l,l\neq i}} + \sum_{\substack{j=l,k\neq i \\ k\neq l,l\neq i}} + \sum_{\substack{k=l,k\neq i \\ k\neq j,j\neq i}} \right) + \sum_{\substack{i,j,k,l \\ i\neq j\neq k\neq l}} .$$

We will have to evaluate each of these fifteen sums separatly. Before that, we will give a useful lemma that we will be using regularly throughout the paper.

Lemma 7. *For any $\alpha \in \mathbb{R}$, we have*

$$\sum_{i\neq j=1}^{n} |i-j|^{\alpha} = \sum_{i,j=0}^{n-1} |i-j|^{\alpha} \leqslant n \sum_{j=0}^{n-1} j^{\alpha}.$$

Proof : The following upperbounds prove this lemma

$$\begin{aligned} \left| \frac{\sum_{i,j=0}^{n-1} |i-j|^{\alpha}}{n\sum_{j=0}^{n-1} j^{\alpha}} \right| = \left| \frac{\sum_{m=0}^{n-1} (n-m) m^{\alpha}}{n\sum_{j=0}^{n-1} j^{\alpha}} \right| \quad &\leq \quad \left| \frac{n\sum_{m=0}^{n-1} m^{\alpha}}{n\sum_{j=0}^{n-1} j^{\alpha}} \right| + \left| \frac{\sum_{m=0}^{n-1} m^{\alpha+1}}{n\sum_{j=0}^{n-1} j^{\alpha}} \right| \\ &\leq \quad 1 + \left| \frac{\sum_{m=0}^{n-1} m^{\alpha+1}}{\sum_{j=0}^{n-1} j^{\alpha+1}} \right| \leq 2. \end{aligned}$$

■

Let's get back to our sums and begin by treating the first one. The first sum can be rewritten as

$$N^{-2} \sum_{i=j=k=l} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$
$$= N^{-2} \sum_{i=1}^{N} \rho(0)^{2r+2+\alpha+\nu+\gamma+\delta} = N^{-2} N \leqslant N^{-1}.$$

For the second sum, we can write

$$N^{-2} \sum_{\substack{i=j=k \\ l\neq i}} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$
$$= N^{-2} \sum_{\substack{i=j=k \\ l\neq i}} \rho(l-i)^{r+1+\gamma+\delta} = N^{-2} \sum_{i\neq l} \rho(l-i)^{q}.$$

At this point, we will use lemma 6 and then lemma 7 to write

$$N^{-2} \sum_{\substack{i=j=k \\ l\neq i}} \rho(k-l)^{r+1} \rho(i-j)^{r+1} \rho(k-i)^{\alpha} \rho(k-j)^{\nu} \rho(l-i)^{\gamma} \rho(l-j)^{\delta}$$
$$\leqslant N^{-2} \sum_{i\neq l=1}^{N} |l-i|^{q(-2\beta+1)} \leqslant N^{-1} \sum_{l=1}^{N-1} l^{q(-2\beta+1)} \leqslant N^{-1}(1 + N^{-2\beta q+q+1})$$
$$\leqslant N^{-1} + N^{-2\beta q+q}.$$

For the third sum, we are in the exact same case, therefore we obtain the same bound $N^{-1}+N^{-2\beta q+q}$. The fourth sum can be handled as follows

$$\begin{aligned}
&N^{-2}\sum_{\substack{i=k=l\\ j\neq i}}\rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}\\
&=N^{-2}\sum_{\substack{i=k=l\\ j\neq i}}\rho(i-j)^{r+1+\nu+\delta}\leqslant N^{-2}\sum_{j\neq i}|i-j|^{(r+1+\nu+\delta)(-2\beta+1)}.
\end{aligned}$$

Note that $r+1+\nu+\delta\geq 1$, so we get

$$\begin{aligned}
&N^{-2}\sum_{\substack{i=k=l\\ j\neq i}}\rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}\\
&\leqslant N^{-2}\sum_{j\neq i}|i-j|^{-2\beta+1}\leqslant N^{-1}\sum_{j=1}^{N-1}j^{-2\beta+1}\leqslant N^{-1}(1+N^{-2\beta+2})\\
&\leqslant N^{-1}+N^{-2\beta+1}.
\end{aligned}$$

For the fifth sum, we are in the exact same case and we obtain the same bound $N^{-1}+N^{-2\beta+1}$. For the sixth sum, we can proceed as follows

$$\begin{aligned}
&N^{-2}\sum_{\substack{i=j,k=l\\ k\neq i}}\rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}\\
&=N^{-2}\sum_{k\neq i}\rho(k-i)^{\alpha+\nu+\gamma+\delta}=N^{-2}\sum_{k\neq i}\rho(k-i)^{2q-2r-2}.
\end{aligned}$$

Recalling that $r\leq q-2\Leftrightarrow 2(q-r-1)\geq 2$, we obtain

$$\begin{aligned}
&N^{-2}\sum_{\substack{i=j,k=l\\ k\neq i}}\rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}\\
&\leqslant N^{-2}\sum_{k\neq i}|k-i|^{(2q-2r-2)(-2\beta+1)}\leqslant N^{-2}\sum_{k\neq i}|k-i|^{-4\beta+2}\leqslant N^{-1}\sum_{k=1}^{N-1}k^{-4\beta+2}\\
&\leqslant N^{-1}+N^{-4\beta+2}.
\end{aligned}$$

We obtain the same bound, $N^{-1}+N^{-4\beta+2}$, for the seventh and eighth sums. For the ninth sum, we have to deal with the following quantity.

$$\begin{aligned}
&N^{-2}\sum_{\substack{i=j,k\neq i\\ k\neq l,l\neq i}}\rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta}\\
&=N^{-2}\sum_{\substack{k\neq i\\ k\neq l,l\neq i}}\rho(k-l)^{r+1}\rho(k-i)^{q-r-1}\rho(l-i)^{q-r-1}.
\end{aligned}$$

For $\sum_{\substack{k\neq i\\ k\neq l,l\neq i}}$, observe that it can be decomposed into

$$\sum_{k>l>i}+\sum_{k>i>l}+\sum_{l>i>k}+\sum_{i>l>k}+\sum_{i>k>l}. \tag{2.30}$$

For the first of the above sums, we can write

$$\begin{aligned}
&N^{-2} \sum_{k>l>i} \rho(k-l)^{r+1}\rho(k-i)^{q-r-1}\rho(l-i)^{q-r-1} \\
&\leqslant N^{-2} \sum_{k>l>i} (k-l)^{(r+1)(-2\beta+1)}(k-i)^{(q-r-1)(-2\beta+1)}(l-i)^{(q-r-1)(-2\beta+1)} \\
&\leqslant N^{-2} \sum_{k>l>i} (k-l)^{q(-2\beta+1)}(l-i)^{(q-r-1)(-2\beta+1)} \quad \text{since } k-i>k-l \\
&= N^{-2} \sum_{k}\sum_{l<k}(k-l)^{q(-2\beta+1)} \sum_{i<l}(l-i)^{(q-r-1)(-2\beta+1)} \\
&\leqslant N^{-2} \sum_{k}\sum_{l<k}(k-l)^{q(-2\beta+1)} \sum_{i<l}(l-i)^{-2\beta+1} \quad \text{since } q-r-1 \geq 1 \\
&\leqslant N^{-2} \sum_{k=1}^{N}\sum_{l=1}^{k-1}(k-l)^{q(-2\beta+1)} \sum_{i=1}^{l-1}(l-i)^{-2\beta+1}.
\end{aligned}$$

Note that $\sum_{l=1}^{k-1}(k-l)^{q(-2\beta+1)} = \sum_{l=1}^{k-1} l^{q(-2\beta+1)}$ and that $\sum_{i=1}^{l-1}(l-i)^{-2\beta+1} = \sum_{i=1}^{l-1} i^{-2\beta+1}$. We can also bound the terms $\sum_{l=1}^{k-1} l^{q(-2\beta+1)}$ (resp. $\sum_{i=1}^{l-1} i^{-2\beta+1}$) from above by $\sum_{l=1}^{N-1} l^{q(-2\beta+1)}$ (resp. $\sum_{i=1}^{N-1} i^{-2\beta+1}$). It follows that

$$\begin{aligned}
&N^{-2} \sum_{k>l>i} \rho(k-l)^{r+1}\rho(k-i)^{q-r-1}\rho(l-i)^{q-r-1} \\
&\leqslant N^{-2} \sum_{k=1}^{N}\sum_{l=1}^{N-1} l^{q(-2\beta+1)} \sum_{i=1}^{N-1} i^{-2\beta+1} \\
&\leqslant N^{-1} \sum_{l=1}^{N-1} l^{q(-2\beta+1)} \sum_{i=1}^{N-1} i^{-2\beta+1} \\
&\leqslant N^{-1}(1+N^{-2\beta q+q+1})(1+N^{-2\beta+2}) \\
&\leqslant N^{-1}+N^{-2\beta+1}+N^{-2\beta q+q}+N^{-2\beta q-2\beta+2}.
\end{aligned}$$

Since $-2\beta+1<0$, $-2\beta q+q<0$ and that $-2\beta q-2\beta+2<0$, it is easy to check that

$$-2\beta q-2\beta+2 < -2\beta q+q < -2\beta+1.$$

Consequently,

$$N^{-2} \sum_{k>l>i} \rho(k-l)^{r+1}\rho(k-i)^{q-r-1}\rho(l-i)^{q-r-1} \leqslant N^{-1}+N^{-2\beta+1}.$$

We obtain the exact same bound $N^{-1}+N^{-2\beta+1}$ for the other terms of the decomposition (2.30) as well as for the tenth, eleventh, twelfth, thirteenth and fourteenth sums by applying the exact same method.

This leaves us with the last (fifteenth) sum. We can decompose $\sum_{\substack{i,j,k,l \\ i\neq j\neq k\neq l}}$ as follows

$$\sum_{k>l>i>j} + \sum_{k>l>j>i} + \ldots \tag{2.31}$$

For the first term, we have

$$\begin{aligned}
& N^{-2} \sum_{k>l>i>j} \rho(k-l)^{r+1}\rho(i-j)^{r+1}\rho(k-i)^{\alpha}\rho(k-j)^{\nu}\rho(l-i)^{\gamma}\rho(l-j)^{\delta} \\
& \leqslant N^{-2} \sum_{k>l>i>j} (k-l)^{q(-2\beta+1)}(i-j)^{(r+1)(-2\beta+1)}(l-i)^{(q-r-1)(-2\beta+1)} \\
& = N^{-2} \sum_{k}\sum_{l<k}(k-l)^{q(-2\beta+1)} \sum_{i<l}(l-i)^{(q-r-1)(-2\beta+1)} \sum_{j<i}(i-j)^{(r+1)(-2\beta+1)} \\
& \leqslant N^{-1} \sum_{l=1}^{N-1} l^{q(-2\beta+1)} \sum_{i=1}^{N-1} i^{(q-r-1)(-2\beta+1)} \sum_{j=1}^{N-1} j^{(r+1)(-2\beta+1)} \\
& \leqslant N^{-1}(1+N^{-2\beta q+q+1})(1+N^{(q-r-1)(-2\beta+1)+1})(1+N^{(r+1)(-2\beta+1)+1}) \\
& \leqslant N^{-1}(1+N^{-2\beta q+q+1})(1+N^{(r+1)(-2\beta+1)+1}+N^{q(-2\beta+1)-(r+1)(-2\beta+1)+1}+N^{q(-2\beta+1)+2}) \\
& \leqslant N^{-1}(1+N^{-2\beta q+q+1})(1+N^{-2\beta+2}+N^{-2\beta+2}+N^{q(-2\beta+1)+2}) \;\text{ since } r+1, q-r-1 \geq 1 \\
& \leqslant N^{-1}(1+N^{-2\beta+2}+N^{q(-2\beta+1)+2}) \\
& \leqslant N^{-1}+N^{-2\beta+1}+N^{q(-2\beta+1)+1}.
\end{aligned}$$

We find the same bound $N^{-1}+N^{-2\beta+1}+N^{q(-2\beta+1)+1}$ for the other terms of the decomposition (2.31).

Finally, by combining all these bounds, we find that

$$\max_{r=1,\ldots,q-1} \mathbf{E}\left(A_r^2\right) \leqslant N^{-2\beta+1}+N^{q(-2\beta+1)+1},$$

and we obtain

$$\mathbf{E}\left(\left(\frac{1}{q}\|DZ_N\|_{\mathcal{H}}^2-1\right)^2\right) \leqslant N^{-2\beta+1}+N^{q(-2\beta+1)+1},$$

which allow us to complete the proof. ■

Remark 3. *1. When $q=2$, $\frac{q}{2q-2}=1$, so the second line of theorem 6 vanishes. If $q>2$, both lines exists and $\frac{q}{2q-2} \underset{q\to+\infty}{\longrightarrow} \frac{1}{2}$.*

2. When $q<(2\beta-1)^{-1}$, the sequence Z_N does not converge in law towards an $\mathcal{N}(0,1)$. It converges (with another normalization) to a Hermite random variable.

3. The results in the above theorem are coherent with those found in [NP09c], Theorem 4.1. Indeed, in [NP09c] one works with $Y_n=B_{n+1}^H-B_n^H$ instead of X_n, where B^H is a fractional Brownian motion. Note that the covariance function $\rho'(m)=\mathbf{E}(Y_0Y_m)$ of Y behaves as m^{2H-2} while, as it follows from Lemma 6, the covariance of X behaves as $m^{-2\beta+1}$. Thus β corresponds to $\frac{3}{2}-H$. It can be seen that Theorem 6 is in concordance with Theorem 4.1 in [NP09c].

2.3.3 Error Bounds in the Non-Central Limit Theorem

We will now turn our attention to the case where $q<(2\beta-1)^{-1}$, where we will use the total variation distance instead of the Kolmogorov distance because that is the distance which appears in a result by Davydov and Martynova [DM89]. This result will be central to our proof of the bounds. Recall that the total variation distance between the probability distributions of two real-valued random variables X and Y is defined by

$$d_{\mathrm{TV}}(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbf{P}(Y \in A) - \mathbf{P}(X \in A)| \tag{2.32}$$

where $\mathcal{B}(\mathbb{R})$ denotes the class of Borel sets of $\mathbb{R}$. We have the following result by Davydov and Martynova [DM89] on the total variation distance between elements of a fixed Wiener chaos.

Theorem 7. *Fix an integer $q \geq 2$ and let $f \in \mathcal{H}^{\odot q} \backslash \{0\}$. Then, for any sequence $\{f_n\}_{n \geq 1} \subset \mathcal{H}^{\odot q}$ converging to f, their exists a constant $c_{q,f}$, depending only on q and f, such that*

$$d_{TV}(I_q(f_n), I_q(f)) \leq c_{q,f} \left\| f_n - f \right\|_{\mathcal{H}^{\odot q}}^{1/q}.$$

We will now use the scaling property of the Brownian motion to introduce a new sequence U_N that has the same law as S_N. Recall that S_N is defined by

$$S_N = \sum_{n=1}^{N} H_q \left(\sum_{i=1}^{\infty} \alpha_i \left(W_{n-i} - W_{n-i-1} \right) \right).$$

Let U_N be defined by

$$U_N = \sum_{n=1}^{N} H_q \left(\sum_{i=1}^{\infty} \alpha_i N^{\frac{1}{2}} \left(W_{\frac{n-i}{N}} - W_{\frac{n-i-1}{N}} \right) \right).$$

Based on the scaling property of the Brownian motion, U_N has the same law as S_N for every fixed N. Recall that Theorem 5 states that

$$h_{q,\beta}^{-1} N^{\beta q - \frac{q}{2} - 1} S_N \underset{N \to +\infty}{\longrightarrow} Z^{(q)}$$

where $Z^{(q)}$ is a Hermite random variable of order q (it is actually the value at time 1 of the Hermite process of order q with self-similarity index

$$\frac{q}{2} - q\beta + 1$$

defined in [CTV09]). Let us first prove the following renormalization result.

Lemma 8. *Let*

$$h_{q,\beta}^2 = \frac{2c_\beta^q}{q!(-2\beta q + q + 1)(-2\beta + q + 2)}. \tag{2.33}$$

Then

$$\mathbf{E} \left(h_{q,\beta}^{-1} N^{\beta q - \frac{q}{2} - 1} S_N \right)^2 \underset{N \to +\infty}{\longrightarrow} 1.$$

Proof : Define $f_N = \sum_{n=1}^{N} f_n^{\otimes n}$. Since $S_N = \frac{1}{q!} I_q(f_N)$ we have

$$\begin{aligned}
&\mathbf{E} \left(h_{q,\beta}^{-1} N^{\beta q - \frac{q}{2} - 1} S_N \right)^2 \\
&= h_{q,\beta}^{-2} \frac{1}{(q!)} N^{2\beta q - q - 2} \sum_{n,m=1}^{N} \rho(|n-m|)^q \\
&= h_{q,\beta}^{-2} \frac{1}{(q!)} N^{2\beta q - q - 2} N \rho(0)^q + 2 h_{q,\beta}^{-2} \frac{1}{(q!)} N^{2\beta q - q - 2} \sum_{n,m=1; n>m}^{N} \rho(n-m)^q \\
&\sim 2 h_{q,\beta}^{-2} \frac{1}{(q!)^2} N^{2\beta q - q - 2} \sum_{n,m=1; n>m}^{N} \rho(n-m)^q
\end{aligned}$$

where for the last equivalence we notice that the diagonal term $h_{q,\beta}^{-2}\frac{1}{(q!)}N^{2\beta q-q-2}N\rho(0)^q$ converges to zero since $q<\frac{1}{2\beta-1}$. Therefore, by using the change of indices $n-m=k$ we can write

$$\begin{aligned}
\mathbf{E}\left(h_{q,\beta}^{-1}N^{\beta q-\frac{q}{2}-1}S_N\right)^2 &= h_{q,\beta}^{-2}\frac{1}{(q!)}N^{2\beta q-q-2}\sum_{n,m=1}^{N}\rho(|n-m|)^q\\
&\sim 2h_{q,\beta}^{-2}\frac{1}{(q!)}N^{2\beta q-q-2}\sum_{k=1}^{N}(N-k)\rho(k)^q\\
&\sim 2h_{q,\beta}^{-2}\frac{c_\beta^q}{(q!)}N^{2\beta q-q-2}\sum_{k=1}^{N}(N-k)k^{-2\beta q+q}
\end{aligned}$$

because, according to Lemma 6, $\rho(k)$ behaves as $c_\beta k^{-2\beta+1}$ when k goes to ∞. Consequently,

$$\mathbf{E}\left(h_{q,\beta}^{-1}N^{\beta q-\frac{q}{2}-1}S_N\right)^2\sim 2h_{q,\beta}^{-2}\frac{c_\beta^q}{q!}\frac{1}{N}\sum_{k=1}^{N}\left(1-\frac{k}{N}\right)\left(\frac{k}{N}\right)^{-2\beta q+q}$$

and this converges to 1 as $N\to\infty$ because $\frac{1}{N}\sum_{k=1}^{N}\left(1-\frac{k}{N}\right)\left(\frac{k}{N}\right)^{-2\beta q+q}$ converges to

$$\int_0^1(1-x)x^{-2\beta q+q}dx=\frac{1}{(-2\beta q+q+1)(-2\beta q+q+2)}.$$

$\blacksquare$

Let Z_N be defined here by

$$Z_N=N^{\beta q-\frac{q}{2}-1}U_N=N^{\beta q-\frac{q}{2}-1}\sum_{n=1}^{N}H_q\left(\sum_{i=1}^{\infty}\alpha_iN^{\frac12}\left(W_{\frac{n-i}{N}}-W_{\frac{n-i-1}{N}}\right)\right).$$

We also know that $h_{q,\beta}^{-1}Z_N\underset{N\to+\infty}{\longrightarrow}Z^{(q)}$ in law (because U_N has the same law as S_N), with $Z^{(q)}$ given by (2.14). Let us give a proper representation of Z_N as an element of the q^{th}-chaos. We have

$$\begin{aligned}
Z_N &= N^{\beta q-\frac{q}{2}-1}\sum_{n=1}^{N}H_q\left(\sum_{i=1}^{\infty}\alpha_iN^{\frac12}\left(W_{\frac{n-i}{N}}-W_{\frac{n-i-1}{N}}\right)\right)\\
&= N^{\beta q-\frac{q}{2}-1}\sum_{n=1}^{N}H_q\left(I_1\left(N^{\frac12}\sum_{i=1}^{\infty}\alpha_i\mathbf{1}_{\left[\frac{n-i-1}{N},\frac{n-i}{N}\right]}\right)\right)\\
&= N^{\beta q-\frac{q}{2}-1}\sum_{n=1}^{N}\frac{1}{q!}I_q\left(\left(N^{\frac12}\sum_{i=1}^{\infty}\alpha_i\mathbf{1}_{\left[\frac{n-i-1}{N},\frac{n-i}{N}\right]}\right)^{\otimes q}\right)\\
&= \frac{1}{q!}I_q\left(N^{\beta q-1}\sum_{n=1}^{N}\left(\sum_{i=1}^{\infty}\alpha_i\mathbf{1}_{\left[\frac{n-i-1}{N},\frac{n-i}{N}\right]}\right)^{\otimes q}\right)\\
&:= \frac{1}{q!}I_q\left(\underbrace{N^{\beta q-1}\sum_{n=1}^{N}g_n^{\otimes q}}_{g_N}\right)
\end{aligned}$$

with $g_n = \sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{[\frac{n-i-1}{N}, \frac{n-i}{N}]}$ and $g_N = N^{\beta q-1} \sum_{n=1}^{N} g_n^{\otimes q} \in \mathcal{H}^{\odot q}$. We will see that $h_{q,\beta}^{-1} Z_N$ converges towards Z in $L^2(\Omega)$, or equivalently that $\left\{ \frac{1}{q!} h_{q,\beta}^{-1} g_N \right\}_{N \geq 1}$ converges in $L^2(\mathbb{R}^{\otimes q}) = \mathcal{H}^{\otimes q}$ to the kernel g of the Hermite random variable (2.15) by computing the following L^2 norm.

$$\mathbf{E}\left(\left| h_{q,\beta}^{-1} Z_N - Z \right|^2 \right) = \mathbf{E}\left(\left| I_q(\frac{1}{q!} h_{q,\beta}^{-1} g_N) - I_q(g) \right|^2 \right) = q! \left\| \frac{1}{q!} h_{q,\beta}^{-1} g_N - g \right\|_{\mathcal{H}^{\otimes q}}^2 .$$

We will now study $\|g_N - g\|_{\mathcal{H}^{\otimes q}}^2$ and establish the rate of convergence of this quantity.

Proposition 6. *We have*

$$\left\| h_{q,\beta}^{-1} \frac{1}{q!} g_N - g \right\|_{\mathcal{H}^{\otimes q}}^2 = \mathcal{O}(N^{2\beta q - q - 1}).$$

In particular the sequence $h_{q,\beta}^{-1} \frac{1}{q!} g_N$ converges in $L^2(\mathbb{R}^{\otimes q})$ as $N \to \infty$ to the kernel of the Hermite process g (2.15).

Proof : We have

$$\begin{aligned}
\|g_N\|_{\mathcal{H}^{\otimes q}}^2 &= N^{2\beta q-2} \sum_{n,k=1}^{N} \langle g_n, g_k \rangle_{\mathcal{H}}^q \\
&= N^{2\beta q-2} \sum_{n,k=1}^{N} \left(\int_{\mathbb{R}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_i \alpha_j \mathbf{1}_{[\frac{n-i-1}{N}, \frac{n-i}{N}]}(u) \mathbf{1}_{[\frac{k-j-1}{N}, \frac{k-j}{N}]}(u) du \right)^q \\
&= N^{2\beta q-2} \sum_{n,k=1}^{N} \left(\sum_{i=1}^{\infty} \alpha_i \alpha_{i+|n-k|} \int_{\frac{n-i-1}{N}}^{\frac{n-i}{N}} du \right)^q \\
&= N^{2\beta q-q-2} \sum_{n,k=1}^{N} \left(\sum_{i=1}^{\infty} i^{-\beta} \left(i + |n-k| \right)^{-\beta} \right)^q . \qquad (2.34)
\end{aligned}$$

In addition, based on the definition of the Hermite process, we have

$$d(q,\beta)^2 q! \, \|g\|_{\mathcal{H}^{\otimes q}}^2 = 1.$$

Let us now compute the scalar product $\langle g_N, g \rangle_{\mathcal{H}^{\otimes q}}$ where g is given by (2.15). It holds that

$$\begin{aligned}
\langle g_N, g \rangle_{\mathcal{H}^{\otimes q}} &= d(q,\beta) N^{\beta q-1} \sum_{n=1}^{N} \langle g_n^{\otimes q}, g \rangle_{\mathcal{H}^{\otimes q}} \\
&= d(q,\beta) N^{\beta q-1} \sum_{n=1}^{N} \int_0^1 \left(\sum_{i \geq 1} \alpha_i \int_{\mathbb{R}} (u-y)_+^{-\beta} 1_{(\frac{n-i-1}{N}, \frac{n-i}{N}]}(y) dy \right)^q du \\
&= d(q,\beta) N^{\beta q-1} \sum_{n=1}^{N} \sum_{k=1}^{N} \int_{\frac{k-1}{N}}^{\frac{k}{N}} \left(\sum_{i \geq 1} \alpha_i \int_{\mathbb{R}} (u-y)_+^{-\beta} 1_{(\frac{n-i-1}{N}, \frac{n-i}{N}]}(y) dy \right)^q du.
\end{aligned}$$

We will now perform the change of variables $u' = (u - \frac{k-1}{N})N$ and $y' = (y - \frac{n-i-1}{N})N$

(renaming the variables by u and y), obtaining

$$\begin{aligned}\langle g_N, g\rangle_{\mathcal{H}^{\otimes q}} &= d(q,\beta)N^{\beta q-1}N^{-q-1}\sum_{n=1}^{N}\sum_{k=1}^{N}\int_0^1\left(\sum_{i\geq 1}\alpha_i\int_0^1\left(\frac{u-y+k-n+i}{N}\right)_+^{-\beta}dy\right)^q du\\ &\sim d(q,\beta)N^{\beta q-q-2}\sum_{n=1}^{N}\sum_{k=1}^{N-1}\left(\sum_{i\geq 1}\alpha_i\left(\frac{k-n+i}{N}\right)_+^{-\beta}\right)^q\end{aligned}$$

where we used the fact that when $N\to\infty$, the quantity $\frac{u-y}{N}$ is negligible. Hence, by eliminating the diagonal term as above,

$$\begin{aligned}\langle g_N, g\rangle_{\mathcal{H}^{\otimes q}} &\sim d(q,\beta)N^{2\beta q-q-2}\sum_{k,n=1;k>n}\left(\sum_{i\geq 1}\alpha_i(i+k-n)^{-\beta}\right)^q\\ &\quad +d(q,\beta)N^{2\beta q-q-2}\sum_{k,n=1;k<n}\left(\sum_{i\geq n-k}\alpha_i(i+k-n)^{-\beta}\right)^q\end{aligned}$$

and by using the change of indices $k-n=l$ in the first summand above and $n-k=l$ in the second summand we observe that

$$\begin{aligned}\langle g_N, g\rangle_{\mathcal{H}^{\otimes q}} &\sim d(q,\beta)N^{2\beta q-q-2}\sum_{l=1}^{N}(N-l)\left(\sum_{i\geq 1}i^{-\beta}(i+l)^{-\beta}\right)^q\\ &\quad +d(q,\beta)N^{2\beta q-q-2}\sum_{l=1}^{N}(N-l)\left(\sum_{i\geq l}i^{-\beta}(i-l)^{-\beta}\right)^q.\end{aligned}\tag{2.35}$$

By summarizing the above estimates (2.34) and (2.35), we establish that

$$\begin{aligned}\left\|h_{q,\beta}^{-1}\frac{1}{q!}g_N-g\right\|^2_{\mathcal{H}^{\otimes q}} &\sim N^{2\beta q-q-1}\Bigg[2h_{q,\beta}^{-2}\frac{1}{(q!)^2}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq 1}i^{-\beta}(i+k)^{-\beta}\right)^q\\ &\quad -2d(q,\beta)h_{q,\beta}^{-1}\frac{1}{q!}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq 1}i^{-\beta}(i+k)^{-\beta}\right)^q\\ &\quad -2d(q,\beta)h_{q,\beta}^{-1}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq k}i^{-\beta}(i-k)^{-\beta}\right)^q\\ &\quad +\frac{1}{d(q,\beta)^2q!}N^{-2\beta q+q+1}\Bigg].\end{aligned}$$

To obtain the conclusion, it suffices to check that the sequence

$$\begin{aligned}a_N &:= 2h_{q,\beta}^{-2}\frac{1}{(q!)^2}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq 1}i^{-\beta}(i+k)^{-\beta}\right)^q\\ &\quad -2d(q,\beta)h_{q,\beta}^{-1}\frac{1}{q!}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq 1}i^{-\beta}(i+k)^{-\beta}\right)^q\\ &\quad -2d(q,\beta)h_{q,\beta}^{-1}\frac{1}{q!}\frac{1}{N}\sum_{k=1}^{N}(N-k)\left(\sum_{i\geq k}i^{-\beta}(i-k)^{-\beta}\right)^q+\frac{1}{q!}N^{-2\beta q+q+1}\end{aligned}$$

is uniformly bounded by a constant with respect to N. Since $d(q,\beta)h_{q,\beta}^{-1} = \frac{1}{q!}h_{q,\beta}^{-2}$, $\sum_{i\geq 1} i^{-\beta}(i+k)^{-\beta} \sim c_\beta k^{-2\beta q+q}$ and

$$\sum_{i\geq k} i^{-\beta}(i-k)^{-\beta} = \sum_{i\geq 1} i^{-\beta}(i+k)^{-\beta}$$

(by the change of notation $i-k=j$), the sequence a_N can be written as

$$a_N \sim \frac{1}{q!}\left(-(-2\beta q+q+1)(-2\beta q+q+2)\frac{1}{N}\sum_{k=1}^{N}(N-k)k^{-2\beta q+q} + N^{-2\beta q+q+1}\right).$$

It is easy to check that

$$\begin{aligned}
N^{-2\beta q+q+1} &= N^{-2\beta q+q+1}(-2\beta q+q+1)(-2\beta q+q+2)\int_0^1 (1-x)x^{-2\beta q+q}dx \\
&= (-2\beta q+q+1)(-2\beta q+q+2)\frac{1}{N}\int_0^N (N-y)y^{-2\beta q+q}dy
\end{aligned}$$

(by the change of variables $xN=y$). Thus,

$$\begin{aligned}
q!a_N &\sim c\frac{1}{N}\sum_{k=1}^{N}\int_{k-1}^{k} dy\left((N-y)y^{-2\beta q+q} - (N-k)k^{-2\beta q+q}\right) \\
&\leq \sum_{k=1}^{N}\int_{k-1}^{k} dy\left|y^{-2\beta q+q} - k^{-2\beta q+q}\right| + \frac{1}{N}\sum_{k=1}^{N}\int_{k-1}^{k} dy\left|y^{-2\beta q+q+1} - k^{-2\beta q+q+1}\right| \\
&\leq \sum_{k=1}^{N}\left((k-1)^{-2\beta q+q} - k^{-2\beta q+q}\right) + \frac{1}{N}\sum_{k=1}^{N}\left(k^{-2\beta q+q+1} - (k-1)^{-2\beta q+q+1}\right)
\end{aligned}$$

and elementary computations show that the terms on the last line above are of order of $N^{-2\beta q+q+1}$. ■

As a consequence of Proposition 6 and of Theorem 7, we obtain

Theorem 8. *Let $q < \frac{1}{2\beta-1}$ and let S_N be given by (2.7).*

$$d_{TV}\left(h_{q,\beta}^{-1}N^{\beta q-\frac{q}{2}-1}S_N, Z^{(q)}\right) \leq C_0(q,\beta)N^{2\beta q-q-1}$$

where $Z^{(q)}$ is given by (2.14), $h_{q,\beta}$ is given by (2.33) and $C_0(q,\beta)$ is a positive constant.

2.4 Application : Hsu-Robbins and Spitzer theorems for moving averages

In this section, we will give an application of the bounds obtained in Theorems 6 and 8. The purpose of the Spitzer theorem for moving averages is to find the asymptotic behavior as $\varepsilon \to 0$ of the sequences

$$f_1(\varepsilon) = \sum_{N\geq 1}\frac{1}{N}\mathbf{P}\left(|S_N| > \varepsilon N\right).$$

when $q > \frac{1}{2\beta-1}$ and

$$f_2(\varepsilon) = \sum_{N\geq 1}\frac{1}{N}\mathbf{P}\left(|S_N| > \varepsilon N^{-2\beta q+q+2}\right).$$

when $q < \frac{1}{2\beta-1}$. The cases of the increments of the fractional Brownian motion were treated in [Tud09]. The same arguments can be applied here. Let us briefly describe the method used to find the limit of $f(\varepsilon)$ as $\varepsilon \to 0$. Let $q > \frac{1}{2\beta-1}$ so the limit of $\sigma^{-1}\frac{1}{\sqrt{N}}S_N$ is a standard normal random variable. We have

$$\begin{aligned}
f_1(\varepsilon) &= \sum_{N\geq 1}\frac{1}{N}\mathbf{P}\left(\sigma^{-1}\frac{1}{\sqrt{N}}|S_N| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) \\
&= \sum_{N\geq 1}\frac{1}{N}\mathbf{P}\left(|Z| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) \\
&\quad + \sum_{N\geq 1}\frac{1}{N}\left[\mathbf{P}\left(\sigma^{-1}\frac{1}{\sqrt{N}}|S_N| > \frac{\varepsilon\sqrt{N}}{\sigma}\right) - \mathbf{P}\left(|Z| > \frac{\varepsilon\sqrt{N}}{\sigma}\right)\right]
\end{aligned}$$

where Z denotes a standard normal random variable. The first summand above was estimated in [Tud09], Lemma 1 while the second summand converges to zero by using the bound in Theorem 6 and the proof of the Proposition 1 in [Tud09]. When $q < \frac{1}{2\beta-1}$, similarly

$$\begin{aligned}
f_2(\varepsilon) &= \sum_{N\geq 1}\frac{1}{N}\mathbf{P}\left(h_{q,\beta}^{-1}N^{\beta q-\frac{q}{2}-1}|S_N| > d_{q,\beta}^{-1}\varepsilon N^{-\beta q+\frac{q}{2}+1}\right) \\
&= \sum_{N\geq 1}\frac{1}{N}\mathbf{P}\left(\left|Z^{(q)}\right| > d_{q,\beta}^{-1}\varepsilon N^{-\beta q+\frac{q}{2}+1}\right) \\
&\quad + \sum_{N\geq 1}\frac{1}{N}\left[\mathbf{P}\left(d_{q,\beta}^{-1}N^{\beta q-\frac{q}{2}-1}|S_N| > d_{q,\beta}^{-1}\varepsilon N^{-\beta q+\frac{q}{2}+1}\right) - \mathbf{P}\left(\left|Z^{(q)}\right| > d_{q,\beta}^{-1}\varepsilon N^{-\beta q+\frac{q}{2}+1}\right)\right]
\end{aligned}$$

with $Z^{(q)}$ a Hermite random variable of order q. The first summand was also estimated in [Tud09], Lemma 1 while the second summand can be handled as in Proposition 2 in [Tud09] and the result in Theorem 8. Hence, we obtain

Proposition 7. *When $q > \frac{1}{2\beta-1}$,*

$$\lim_{\varepsilon\to 0}\frac{1}{-\log(\varepsilon)}f_1(\varepsilon) = 2$$

and when $q < \frac{1}{2\beta-1}$ then

$$\lim_{\varepsilon\to 0}\frac{1}{-\log(\varepsilon)}f_2(\varepsilon) = \frac{1}{1+\frac{q}{2}-\beta q}.$$

It is also possible to give Hsu-Robbins type results, meaning to find the asymptotic behavior as $\varepsilon \to 0$ of

$$g_1(\varepsilon) = \sum_{N\geq 1}\mathbf{P}\left(|S_N| > \varepsilon N\right)$$

when $q > \frac{1}{2\beta-1}$ and

$$g_2(\varepsilon) = \sum_{N\geq 1}\mathbf{P}\left(|S_N| > \varepsilon N^{-2\beta q+q+2}\right)$$

when $q < \frac{1}{2\beta-1}$. This also follows from Section 4 in [Tud09] and Theorems 6 and 8.

Proposition 8. *When* $q > \frac{1}{2\beta-1}$,

$$\lim_{\varepsilon\to 0}(\sigma_{q,\beta}^{-1}\varepsilon)^2 g_1(\varepsilon) = 1 = \mathbf{E}\left(Z^2\right)$$

and when $q < \frac{1}{2\beta-1}$ *then*

$$\lim_{\varepsilon\to 0}(h_{q\beta}^{-1}\varepsilon)^{\frac{1}{1+\frac{q}{2}-\beta q}} g_2(\varepsilon) = \mathbf{E}\left|Z^{(q)}\right|^{\frac{1}{1+\frac{q}{2}-\beta q}} .$$

Chapitre 3

Cramér theorems for Gamma random variables

Solesne Bourguin * and Ciprian A. Tudor † ‡
Université Paris 1 and *Université Lille 1*

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Abstract

In this paper we discuss the following problem : given a random variable $Z = X + Y$ with Gamma law such that X and Y are independent, we want to understand if then X and Y *each* follow a Gamma law. This is related to Cramér's theorem which states that if X and Y are independent then $Z = X + Y$ follows a Gaussian law if and only if X *and* Y follow a Gaussian law. We prove that Cramér's theorem is true in the Gamma context for random variables leaving in a Wiener chaos of fixed order but the result is not true in general. We also give an asymptotic variant of our result.

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3.1 Introduction

Cramér's theorem (see [Cra36]) says that the sum of two independent random variables is Gaussian if and only if each summand is Gaussian. One direction is elementary to prove, that is, given two independent random variables with Gaussian distribution, then their sum follows a Gaussian distribution. The second direction is less trivial and its proof requires powerful results from complex analysis (see [Cra36]).
In this paper, we treat the same problem for Gamma distributed random variables. A Gamma random variable, denoted usually by $\Gamma(a, \lambda)$, is a random variable with probability

∗. SAMM, Université de Paris 1 Panthéon-Sorbonne, 90, rue de Tolbiac, 75634, Paris, France. Email : solesne.bourguin@univ-paris1.fr

†. Laboratoire Paul Painlevé, Université de Lille 1, F-59655 Villeneuve d'Ascq, France. Email : tudor@math.univ-lille1.fr

‡. Associate member of the team Samm, Université de Paris 1 Panthéon-Sorbonne.

density function given by $f_{a,\lambda}(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}$ if $x > 0$ and $f_{a,\lambda}(x) = 0$ otherwise. The parameters a and λ are strictly positive and Γ denotes the usual Gamma function.
It is well known that if $X \sim \Gamma(a,\lambda)$ and $Y \sim \Gamma(b,\lambda)$ and X is independent of Y, then $X+Y$ follows the law $\Gamma(a+b,\lambda)$. The purpose of this paper is to understand the converse implication, i.e. whether or not (or under what conditions), if X and Y are two independent random variables such that $X+Y \sim \Gamma(a+b,\lambda)$ and $\mathbf{E}(X) = \mathbf{E}(\Gamma(a,\lambda)), \mathbf{E}(X^2) = \mathbf{E}(\Gamma(a,\lambda)^2)$ and $\mathbf{E}(Y) = \mathbf{E}(\Gamma(b,\lambda)), \mathbf{E}(Y^2) = \mathbf{E}(\Gamma(b,\lambda)^2)$, it holds that $X \sim \Gamma(a,\lambda)$ and $Y \sim \Gamma(b,\lambda)$.
We will actually focus our attention on the so-called centered Gamma distribution $F(\nu)$. We will call 'centered Gamma' the random variables of the form

$$F(\nu) \overset{\text{Law}}{=} 2G(\nu/2) - \nu, \quad \nu > 0,$$

where $G(\nu/2) := F(\nu/2, 1)$ has a Gamma law with parameters $\nu/2, 1$. This means that $\Gamma(\nu/2,1)$ is a (a.s. strictly positive) random variable with density $g(x) = \frac{x^{\frac{\nu}{2}-1} e^{-x}}{\Gamma(\nu/2)} \mathbf{1}_{(0,\infty)}(x)$. The characteristic function of the law $F(\nu)$ is given by

$$\mathbf{E}\left(e^{i\lambda F(\nu)}\right) = \left(\frac{e^{-i\lambda}}{\sqrt{1-2i\lambda}}\right)^{\nu}, \quad \lambda \in \mathbb{R}. \tag{3.1}$$

We will find the following answer : if X and Y and two independent random variables, each leaving in a Wiener chaos of fixed order (and these orders are allowed to be different) then the fact that the sum $X+Y$ follows a centered Gamma distribution implies that X and Y each follow a Gamma distribution. On the other hand, for random variables having an infinite Wiener-Itô chaos decomposition, the result is not true even in very particular cases (for so-called strongly independent random variables). We construct a counter-example to illustrate this fact.
Our tools are based on a criterium given in [NP09c] to characterize the random variables with Gamma distribution in terms of Malliavin calculus.
Our paper is structured as follows. Section 2 contains some notations and preliminaries. In Section 3 we prove the Cramér theorem for Gamma distributed random variables in Wiener chaos of finite orders and we also give an asymptotic version of this result. In Section 4 we show that the result does not hold in the general case.

3.2 Some notations and definitions

Let $(W_t)_{t\in T}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2(T^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W. The basic references are the monographs [Mal97] or [Nua06]. Let $f \in \mathcal{S}_n$ be an elementary function with n variables that can be written as $f = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} 1_{A_{i_1} \times \ldots \times A_{i_n}}$ where the coefficients satisfy $c_{i_1,\ldots,i_n} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}(T)$ are pairwise disjoint. For such a step function f we define

$$I_n(f) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} W(A_{i_1}) \ldots W(A_{i_n})$$

where we put $W(A) = \int_0^1 1_A(s) dW_s$. It can be seen that the application I_n constructed above from $\mathcal{S}_n$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}_n$ in the sense

$$\mathbf{E}(I_n(f) I_m(g)) = n! \langle f, g \rangle_{L^2(T^n)} \text{ if } m = n \tag{3.2}$$

and

$$\mathbf{E}\left(I_n(f)I_m(g)\right) = 0 \text{ if } m \neq n.$$

Since the set $\mathcal{S}_n$ is dense in $L^2(T^n)$ for every $n \geq 1$ the mapping I_n can be extended to an isometry from $L^2(T^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension.
It also holds that $I_n(f) = I_n(\tilde{f})$ where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!}\sum_{\sigma} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

σ running over all permutations of $\{1, ..., n\}$. We will need the general formula for calculating products of Wiener chaos integrals of any orders m, n for any symmetric integrands $f \in L^2(T^m)$ and $g \in L^2(T^n)$, which is

$$I_m(f)I_n(g) = \sum_{\ell=0}^{m\wedge n} \ell! \binom{m}{\ell}\binom{n}{\ell} I_{m+n-2\ell}(f \otimes_\ell g) \tag{3.3}$$

where the contraction $f \otimes_\ell g$ is defined by

$$\begin{aligned}
&(f \otimes_\ell g)(s_1, \ldots, s_{m-\ell}, t_1, \ldots, t_{n-\ell}) \\
&= \int_{T^{m+n-2\ell}} f(s_1, \ldots, s_{m-\ell}, u_1, \ldots, u_\ell) g(t_1, \ldots, t_{n-\ell}, u_1, \ldots, u_\ell) du_1 \ldots du_\ell.
\end{aligned} \tag{3.4}$$

Note that the contraction $(f \otimes_\ell g)$ is an element of $L^2(T^{m+n-2\ell})$ but it is not necessarily symmetric. We will denote its symmetrization by $(f \tilde{\otimes}_\ell g)$.
We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n \geq 0} I_n(f_n) \tag{3.5}$$

where $f_n \in L^2(T^n)$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}\left(F\right)$.
We denote by D the Malliavin derivative operator that acts on smooth functionals of the form $F = g(W(\varphi_1), \ldots, W(\varphi_n))$ (here g is a smooth function with compact support and $\varphi_i \in L^2(T)$ for $i = 1, .., n$)

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_n))\varphi_i.$$

We can define the i-th Malliavin derivative $D^{(i)}$ iteratively. The operator $D^{(i)}$ can be extended to the closure $\mathbb{D}^{p,2}$ of smooth functionals with respect to the norm

$$\|F\|_{p,2}^2 = \mathbf{E}\left(F^2\right) + \sum_{i=1}^{p} \mathbf{E}\left(\|D^i F\|_{L^2(T^i)}^2\right).$$

The adjoint of D is denoted by δ and is called the divergence (or Skorohod) integral. Its domain $\mathrm{Dom}(\delta)$ coincides with the class of stochastic processes $u \in L^2(\Omega \times T)$ such that

$$\left|\mathbf{E}\left(\langle DF, u\rangle\right)\right| \leq c\|F\|_2$$

for all $F \in \mathbb{D}^{1,2}$ and $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the duality relationship

$$\mathbf{E}(F\delta(u)) = \mathbf{E}\left(\langle DF, u\rangle\right).$$

For adapted integrands, the divergence integral coincides with the classical Itô integral. Let L be the Ornstein-Uhlenbeck operator defined on $\mathrm{Dom}(L) = \mathbb{D}^{2,2}$. We have

$$LF = -\sum_{n\geq 0} nI_n(f_n)$$

if F is given by (5.3). There exists a connection between δ, D and L in the sense that a random variable F belongs to the domain of L if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}(\delta)$ and then $\delta DF = -LF$. Let us consider a multiple stochastic integral $I_q(f)$ with symmetric kernel $f \in L^2(T^q)$. We denote the Malliavin derivative of $I_q(f)$ by $DI_q(f)$. We have

$$D_\theta I_q(f) = qI_{q-1}(f^{(\theta)}),$$

where $f^{(\theta)} = f(t_1, ..., t_{q-1}, \theta)$ is the $(q-1)^{\text{th}}$ order kernel obtained by parametrizing the q^{th} order kernel f by one of the variables.
For any random variable $X, Y \in \mathbb{D}^{1,2}$ we use the following notations

$$G_X = \langle DX, -DL^{-1}X\rangle_{L^2(T)}$$

and

$$G_{X,Y} = \langle DX, -DL^{-1}Y\rangle_{L^2(T)}.$$

The following facts are key points in our proofs :

Fact 1 : Let $X = I_{q_1}(f)$ and $Y = I_{q_2}(g)$ where $f \in L^2(T^{q_1})$ and $g \in L^2(T^{q_2})$ are symmetric functions. Then X and Y are independent if and only if (see [ÜZ89])

$$f \otimes_1 g = 0 \text{ a.e. on } T^{q_1+q_2-2}.$$

Fact 2 : Let $X = I_q(f)$ with $f \in L^2(T^q)$ symmetric. Assume that $\mathbf{E}\left(X^2\right) = \mathbf{E}(F(\nu)^2) = 2\nu$. Then X follows a centered Gamma law $F(\nu)$ with $\nu > 0$ if and only if (see [NP09a])

$$\|DX\|^2_{L^2(T)} - 2qX - 2q\nu = 0 \text{ almost surely.}$$

Fact 3 : Let $(f_k)_{k\geq 1}$ be a sequence in $L^2(T^q)$ such that $\mathbf{E}\left(I_q(f_k)^2\right) \underset{k\to+\infty}{\longrightarrow} 2\nu$. Then the sequence $X_k = I_q(f_k)$ converges in distribution, as $k \to \infty$, to a Gamma law, if and only if (see [NP09a])

$$\|DX_k\|^2_{L^2(T)} - 2qX_k - 2q\nu \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } L^2(\Omega).$$

Remark: In this particular paper, we will restrict ourselves to an underlying Hilbert space (to the Wiener process we will be working with in the upcoming sections) of the form $\mathfrak{H} = L^2(T)$ for the sake of simplicity. However, all the results presented in the upcoming sections remain valid on a more general separable Hilbert space as the underlying space.

3.3 (Asymptotic) Cramér theorem for multiple integrals

In this section, we will prove Cramér's theorem for random variables living in fixed Wiener chaoses. More precisely, our context is as follows : we assume that $X = I_{q_1}(f)$ and $Y = I_{q_2}(h)$ and X, Y are independent. We also assume that $\mathbf{E}\left(X^2\right) = \mathbf{E}\left(F(\nu_1)^2\right) = 2\nu_1$ and $\mathbf{E}\left(Y^2\right) = \mathbf{E}\left(F(\nu_2^2)\right) = 2\nu_2$. Here ν, ν_1, ν_2 denotes three strictly positive numbers such that

$\nu_1 + \nu_2 = \nu$. We assume that $X+Y$ follows a Gamma law $F(\nu)$ and we will prove that $X \sim F(\nu_1)$ and $Y \sim F(\nu_2)$.
Let us first give the two following auxiliary lemmas that will be useful throughout the paper.

Lemma 9. *Let $q_1, q_2 \geq 1$ be integers, and let $X = I_{q_1}(f)$ and $Y = I_{q_2}(h)$, where $f \in L^2(T^{q_1})$ and $h \in L^2(T^{q_2})$ are symmetric functions. Assume moreover that X and Y are independent. Then, we have $DX \perp DY$, $X \perp DY$ and $Y \perp DX$.*

Proof : From Fact 1 in Section 2, $f \otimes_1 h = 0$ a.e on $T^{q_1+q_2-2}$ and by extension $f \otimes_r h = 0$ a.e on $T^{q_1+q_2-2r}$ for every $1 \leq r \leq q_1 \wedge q_2$. We will now prove that for every $\theta, \psi \in T$, we also have $f^{(\theta)} \otimes_1 h^{(\psi)} = 0$ a.e on $T^{q_1+q_2-4}$, $f^{(\theta)} \otimes_1 h = 0$ a.e on $T^{q_1+q_2-3}$ and $f \otimes_1 h^{(\psi)} = 0$ a.e. on $T^{q_1+q_2-3}$. Indeed, we have

$$\begin{aligned}\left(f^{(\theta)} \otimes_1 h^{(\psi)}\right)(t_1, \ldots, t_{q_1-2}, s_1, \ldots, s_{q_2-2}) &= \int_T f(t_1, ..., t_{q_1-2}, u, \theta) h(s_1, ..., s_{q_2-2}, u, \psi) du \\ &= 0\end{aligned}$$

as a particular case of $f \otimes_1 h = 0$ a.e.. By extension, we also have $f^{(\theta)} \otimes_r h^{(\psi)} = 0$ for $1 \leq r \leq (q_1-1) \wedge (q_2-1)$. Similarly,

$$\begin{aligned}\left(f^{(\theta)} \otimes_1 h\right)(t_1, \ldots, t_{q_1-2}, s_1, \ldots, s_{q_2-1}) &= \int_T f(t_1, ..., t_{q_1-2}, u, \theta) h(s_1, \ldots, s_{q_2-1}, u) du. \\ &= 0 \end{aligned} \tag{3.6}$$

and clearly $f^{(\theta)} \otimes_r h = 0$ for $1 \leq r \leq (q_1-1) \wedge q_2$. Given the symmetric roles played by f and h, we also have $f \otimes_1 h^{(\psi)} = 0$ and then $f \otimes_r h^{(\psi)} = 0$ for $1 \leq r \leq q_1 \wedge (q_2-1)$.
Let us now prove that $DX \perp DY$. Since for every $\theta, \psi \in T$, $D_\theta X = q_1 I_{q_1}(f^{(\theta)})$ and $D_\psi Y = q_2 I_{q_2-1}(h^{(\psi)})$, it suffices to show that the random variables $I_{q_1}(f^{(\theta)})$ and $I_{q_2-1}(h^{(\psi)})$ are independent. To do this, we will use the criterium for the independence of multiple integrals given in [ÜZ89]. We need to check that $f^{(\theta)} \otimes_1 h^{(\psi)} = 0$ a.e. on $T^{q_1+q_2-4}$ and this follows from above.
It remains to prove that $X \perp DY$ and $DX \perp Y$. Given the symmetric roles played by X and Y, we will only prove that $X \perp DY$. That is equivalent to the independence of the random variables $I_{q_1-1}(f^{(\theta)})$ and $I_{q_2}(h)$ for every $\theta \in T$, which follows from [ÜZ89] (see Fact 1 in Section 2) and (3.6). Thus, we have $X \perp DY$ and $DX \perp Y$. ■

Let us recall the following definition (see [Tud11]).

Definition 4. *Two random variables $X = \sum_{n \geq 0} I_n(f_n)$ and $Y = \sum_{m \geq 0} I_m(h_m)$ are called strongly independent if for every $m, n \geq 0$, the random variables $I_n(f_n)$ and $I_m(h_m)$ are independent.*

We have the following lemma about strongly independent random variables.

Lemma 10. *Let $X = \sum_{n \geq 0} I_n(f_n)$ and $Y = \sum_{m \geq 0} I_m(h_m)$ ($f_n \in L^2(T^n), h_m \in L^2(T^m)$ symmetric for every $n, m \geq 1$) be two centered random variables in the space $\mathbb{D}^{1,2}$. Then, if X and Y are strongly independent, we have*

$$\langle DX, -DL^{-1}Y \rangle_{L^2(T)} = \langle DY, -DL^{-1}X \rangle_{L^2(T)} = 0.$$

Proof : We have, for every $\theta \in T$,

$$D_\theta X = \sum_{n \geq 1} n I_{n-1}(f_n^{(\theta)}) \text{ and } -D_\theta L^{-1} Y = \sum_{m \geq 1} I_{m-1}(h_m^{(\theta)}).$$

Therefore, we can write

$$\begin{aligned}
&\langle DX, -DL^{-1}Y\rangle_{L^2(T)} \\
&= \sum_{n,m\geq 1} n\int_T I_{n-1}(f_n(t_1,...,t_{n-1},\theta))I_{m-1}(h_m(t_1,...,t_{m-1},\theta))d\theta \\
&= \sum_{n,m\geq 1} n\int_T \sum_{r=0}^{(n-1)\wedge(m-1)} r!\binom{n-1}{r}\binom{m-1}{r} I_{n+m-2r-2}(f_n^{(\theta)}\otimes_r h_m^{(\theta)})d\theta.
\end{aligned}$$

The strong independence of X and Y gives us that $f_n^{(\theta)}\otimes_r h_m^{(\theta)} = 0$ for every $1\leq r\leq (n-1)\wedge(m-1)$. Thus, we obtain

$$\langle DX, -DL^{-1}Y\rangle_{L^2(T)} = \sum_{n,m\geq 1} n\int_T I_{n+m-2}(f_n^{(\theta)}\otimes h_m^{(\theta)})d\theta.$$

Using a Fubini type result, we can write

$$\begin{aligned}
\langle DX, -DL^{-1}Y\rangle_{L^2(T)} &= \sum_{n,m\geq 1} nI_{n+m-2}(\int_T f_n^{(\theta)}\otimes h_m^{(\theta)}d\theta) \\
&= \sum_{n,m\geq 1} nI_{n+m-2}(f_n\otimes_1 h_m).
\end{aligned}$$

Again, the strong independence of X and Y gives us that $f_n\otimes_1 h_m = 0$ a.e and we finally obtain $\langle DX, -DL^{-1}Y\rangle_{L^2(T)} = 0$, and similarly $\langle DY, -DL^{-1}X\rangle_{L^2(T)} = 0$. ∎

Let us first remark that the Cramér theorem holds for random variables in the same Wiener chaos of fixed order.

Proposition 9. *Let $X = I_m(f)$ and $Y = I_m(h)$ with $m\geq 2$ fixed and f,h symmetric functions in $L^2(T^m)$. Then $X+Y = I_m(f+h)$. Fix $\nu_1,\nu_2,\nu > 0$ such that $\nu_1+\nu_2 = \nu$. Assume that $X+Y$ follows the law $F(\nu)$ and X is independent of Y. Also suppose that $\mathbf{E}(X^2) = \mathbf{E}(F(\nu_1)^2) = 2\nu_1$ and $\mathbf{E}(Y^2) = \mathbf{E}(F(\nu_2)^2) = 2\nu_2$. Then $X\sim F(\nu_1)$ and $Y\sim F(\nu_2)$.*

Proof : By a result in [NP09a] (see Fact 2 in Section 2), $X+Y$ follows the law $F(\nu)$ is equivalent to

$$||DI_m(f+h)||^2_{L^2(T)} - 2mI_m(f+h) - 2m\nu = 0 \text{ a.s. }.$$

On the other hand

$$\begin{aligned}
&\mathbf{E}\left(||DI_m(f+h)||^2_{L^2(T)} - 2mI_m(f+h) - 2m\nu\right)^2 \\
=\ &\mathbf{E}\Big(\Big(||DI_m(f)||^2_{L^2(T)} + ||DI_m(h)||^2_{L^2(T)} + 2\langle DI_m(f), DI_m(h)\rangle_{L^2(T)} \\
&-2mI_m(f) - 2mI_m(h) - 2m(\nu_1+\nu_2)\Big)^2\Big) \\
=\ &\mathbf{E}\left(\left(||DI_m(f)||^2_{L^2(T)} - 2mI_m(f) - 2m\nu_1\right)^2\right) \\
&+\mathbf{E}\left(\left(||DI_m(h)||^2_{L^2(T)} - 2mI_m(h) - 2m\nu_2\right)^2\right) \\
&+\mathbf{E}\left(\left(||DI_m(f)||^2_{L^2(T)} - 2mI_m(f) - 2m\nu_1\right)\left(||DI_m(h)||^2_{L^2(T)} - 2mI_m(h) - 2m\nu_2\right)\right).
\end{aligned}$$

Above we used the fact that $\langle DI_m(f), DI_m(h)\rangle_{L^2(T)} = 0$ as a consequence of Lemma 9. It is also easy to remark that, from Lemma 9

$$\begin{aligned} & \mathbf{E}\left(\left(||DI_m(f)||^2_{L^2(T)} - 2mI_m(f) - 2m\nu_1\right)\left(||DI_m(h)||^2_{L^2(T)} - 2mI_m(h) - 2m\nu_2\right)\right) \\ = \; & \mathbf{E}\left(||DI_m(f)||^2_{L^2(T)} - 2mI_m(f) - 2m\nu_1\right)\mathbf{E}\left(||DI_m(h)||^2_{L^2(T)} - 2mI_m(h) - 2m\nu_2\right) = 0. \end{aligned}$$

We will obtain that

$$\begin{aligned} & \mathbf{E}\left(\left(||DI_m(f)||^2_{L^2(T)} - 2mI_m(f) - 2m\nu_1\right)^2\right) \\ & = \mathbf{E}\left(\left(||DI_m(h)||^2_{L^2(T)} - 2mI_m(h) - 2m\nu_2\right)^2\right) = 0 \end{aligned}$$

and consequently $X \sim F(\nu_1)$ and $Y \sim F(\nu_2)$. ■

Remark 4. *Using Fact 3 in Section 2, an asymptotic variant of the above result can be stated. We will state it here because it is a particular case of Theorem 12 proved later in our paper.*

Theorem 1.2 in [NP09a] gives a characterization of (asymptotically) centered Gamma random variable which are given by a multiple Wiener-Itô integral. There is not such a characterization for random variable leaving in a finite or infinite sum of Wiener chaos; only an upper bound for the distance between the law of a random variable in $\mathbb{D}^{1,2}$ and the Gamma distribution has been proven in [NP09c], Theorem 3.11. It turns out, that for the case of a sum of independent multiple integrals, it is possible to characterize the relation between its distribution and the Gamma distribution. We will prove this fact in the following theorem.

Theorem 9. *Fix $\nu_1, \nu_2, \nu > 0$ such that $\nu_1 + \nu_2 = \nu$ and let $F(\nu)$ be a real-valued random variable with characteristic function given by (3.1). Fix two even integers $q_1 \geq 2$ and $q_2 \geq 2$. For any symmetric kernels $f \in L^2(T^{q_1})$ and $h \in L^2(T^{q_2})$ such that*

$$\mathbf{E}\left(I_{q_1}(f)^2\right) = q_1!\,\|f\|^2_{L^2(T^{q_1})} = 2\nu_1 \quad \textit{and} \quad \mathbf{E}\left(I_{q_2}(h)^2\right) = q_2!\,\|h\|^2_{L^2(T^{q_2})} = 2\nu_2, \qquad (3.7)$$

and such that $X = I_{q_1}(f)$ and $Y = I_{q_2}(h)$ are independent, define the random variable

$$Z = X + Y = I_{q_1}(f) + I_{q_2}(h).$$

Under those conditions, the following two conditions are equivalent :

(i) $\mathbf{E}\left(\left(2\nu + 2Z - \langle DZ, -DL^{-1}Z\rangle_{L^2(T)}\right)^2\right) = 0$, *where D is the Malliavin derivative operator and L is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup;*

(ii) $Z \overset{\text{Law}}{=} F(\nu)$;

Proof : *Proof of $(ii) \to (i)$.* Suppose that $Z \sim F(\nu)$. We easily obtain that

$$\mathbf{E}\left(Z^3\right) = \mathbf{E}\left(F(\nu)^3\right) = 8\nu \;\text{ and }\; \mathbf{E}\left(Z^4\right) = \mathbf{E}\left(F(\nu)^4\right) = 12\nu^2 + 48\nu. \qquad (3.8)$$

Consequently,

$$\mathbf{E}\left(Z^4\right) - 12\mathbf{E}\left(Z^3\right) = \mathbf{E}\left(F(\nu)^4\right) - 12\mathbf{E}\left(F(\nu)^3\right) = 12\nu^2 - 48\nu. \qquad (3.9)$$

Then we will use the fact that for every multiple integral $I_q(f)$

$$\mathbf{E}\left(I_q(f)^3\right) = q!(q/2)!\binom{q}{q/2}^2 \left\langle f, f\widetilde{\otimes}_{q/2} f\right\rangle_{L^2(T^q)}. \tag{3.10}$$

and

$$\begin{aligned}\mathbf{E}\left(I_q(f)^4\right) &= 3\left[q!\,\|f\|^2_{L^2(T^q)}\right]^2 \\ &+\frac{3}{q}\sum_{p=1}^{q-1} q^2(p-1)!\binom{q-1}{p-1}^2 p!\binom{q}{p}^2(2q-2p)!\,\|f\widetilde{\otimes}_p f\|^2_{L^2(T^{2(q-p)})}\end{aligned} \tag{3.11}$$

We will now compute $\mathbf{E}\left(Z^3\right)$, $\mathbf{E}\left(Z^4\right)$ and $\mathbf{E}\left(Z^4\right) - 12\mathbf{E}\left(Z^3\right)$ by using the above two relations (3.10) and (3.11). We have $Z^2 = (I_{q_1}(f)+I_{q_2}(h))^2 = I_{q_1}(f)^2+I_{q_2}(h)^2+2I_{q_1}(f)I_{q_2}(h)$ and thus, by using the independence between $I_{q_1}(f)$ and $I_{q_2}(h)$,

$$\mathbf{E}\left(Z^3\right) = \mathbf{E}\left(I_{q_1}(f)^3\right) + \mathbf{E}\left(I_{q_2}(h)^3\right).$$

Using relation (3.10), we can write

$$\begin{aligned}\mathbf{E}\left(Z^3\right) &= q_1!(q_1/2)!\binom{q_1}{q_1/2}^2\left\langle f, f\widetilde{\otimes}_{q_1/2} f\right\rangle_{L^2(T^{q_1})} \\ &\quad + q_2!(q_2/2)!\binom{q_2}{q_2/2}^2\left\langle h, h\widetilde{\otimes}_{q_2/2} h\right\rangle_{L^2(T^{q_2})}.\end{aligned} \tag{3.12}$$

For $\mathbf{E}\left(Z^4\right)$, we combine relations (3.7) and (3.11) with the independence between $I_{q_1}(f)$ and $I_{q_2}(h)$ to obtain

$$\begin{aligned}\mathbf{E}\left(Z^4\right) &= \mathbf{E}\left(Z^2Z^2\right) = \mathbf{E}\left(I_{q_1}(f)^4\right) + \mathbf{E}\left(I_{q_2}(h)^4\right) + 6\mathbf{E}\left(I_{q_1}(f)^2 I_{q_2}(h)^2\right) \\ &= 3\left[q_1!\,\|f\|^2_{L^2(T^{q_1})}\right]^2 \\ &\quad +\frac{3}{q_1}\sum_{p=1}^{q_1-1} q_1^2(p-1)!\binom{q_1-1}{p-1}^2 p!\binom{q_1}{p}^2(2q_1-2p)!\,\|f\widetilde{\otimes}_p f\|^2_{L^2(T^{2(q_1-p)})} \\ &\quad +3\left[q_2!\,\|h\|^2_{L^2(T^{q_2})}\right]^2 \\ &\quad +\frac{3}{q_2}\sum_{p=1}^{q_2-1} q_2^2(p-1)!\binom{q_2-1}{p-1}^2 p!\binom{q_2}{p}^2(2q_2-2p)!\,\|h\widetilde{\otimes}_p h\|^2_{L^2(T^{2(q_2-p)})} \\ &\quad +24\nu_1\nu_2.\end{aligned}$$

Using the fact that $q_1!\,\|f\|^2_{L^2(T^{q_1})} = 2\nu_1$ and $q_2!\,\|h\|^2_{L^2(T^{q_2})} = 2\nu_2$, we can write

$$
\begin{aligned}
&\mathbf{E}\left(Z^4\right) - 12\mathbf{E}\left(Z^3\right)\\
&= 12\nu_1^2 + 12\nu_2^2 - 48\nu_1 - 48\nu_2 + 24\nu_1\nu_2\\
&+\frac{3}{q_1}\sum_{p=1,p\neq q_1/2}^{q_1-1} q_1^2(p-1)!\binom{q_1-1}{p-1}^2 p!\binom{q_1}{p}^2 (2q_1-2p)!\,\|f\widetilde{\otimes}_p f\|^2_{L^2(T^{2(q_1-p)})}\\
&+\frac{3}{q_2}\sum_{p=1,p\neq q_2/2}^{q_2-1} q_2^2(p-1)!\binom{q_2-1}{p-1}^2 p!\binom{q_2}{p}^2 (2q_2-2p)!\,\|h\widetilde{\otimes}_p h\|^2_{L^2(T^{2(q_2-p)})}\\
&+24q_1!\,\|f\|^2_{L^2(T^{q_1})} + 3q_1(q_1/2-1)!\binom{q_1-1}{q_1/2-1}^2 (q_1/2)!\binom{q_1}{q_1/2}^2 q_1!\left\|f\widetilde{\otimes}_{q_1/2}f\right\|^2_{L^2(T^{q_1})}\\
&+24q_2!\,\|h\|^2_{L^2(T^{q_2})} + 3q_2(q_2/2-1)!\binom{q_2-1}{q_2/2-1}^2 (q_2/2)!\binom{q_2}{q_2/2}^2 q_2!\left\|h\widetilde{\otimes}_{q_2/2}h\right\|^2_{L^2(T^{q_2})}\\
&-12q_1!(q_1/2)!\binom{q_1}{q_1/2}^2 \left\langle f, f\widetilde{\otimes}_{q_1/2}f\right\rangle_{L^2(T^{q_1})}\\
&-12q_2!(q_2/2)!\binom{q_2}{q_2/2}^2 \left\langle h, h\widetilde{\otimes}_{q_2/2}h\right\rangle_{L^2(T^{q_2})}. \qquad (3.13)
\end{aligned}
$$

Recall that $\nu = \nu_1 + \nu_2$ and note that $12\nu_1^2 + 12\nu_2^2 - 48\nu_1 - 48\nu_2 + 24\nu_1\nu_2 = 12\nu^2 - 48\nu$. Also note that

$$
\begin{aligned}
&24q_1!\,\|f\|^2_{L^2(T^{q_1})} + 3q_1(q_1/2-1)!\binom{q_1-1}{q_1/2-1}^2 (q_1/2)!\binom{q_1}{q_1/2}^2 q_1!\left\|f\widetilde{\otimes}_{q_1/2}f\right\|^2_{L^2(T^{q_1})}\\
&-12q_1!(q_1/2)!\binom{q_1}{q_1/2}^2 \left\langle f, f\widetilde{\otimes}_{q_1/2}f\right\rangle_{L^2(T^{q_1})}\\
=\ &\frac{3}{2}\frac{(q_1!)^5}{((q_1/2)!)^6}\left\|f\widetilde{\otimes}_{q_1/2}f - c_{q_1}f\right\|^2_{L^2(T^{q_1})},
\end{aligned}
$$

where c_{q_1} is defined by $c_{q_1} = \frac{1}{(q_1/2)!\binom{q_1-1}{q_1/2-1}^2} = \frac{4}{(q_1/2)!\binom{q_1}{q_1/2}^2}$ and a similar relation holds for the function h with q_2, c_{q_2} instead of q_1, c_{q_1} respectively, where $c_{q_2} = \frac{1}{(q_2/2)!\binom{q_2-1}{q_2/2-1}^2} = \frac{4}{(q_2/2)!\binom{q_2}{q_2/2}^2}$.

$$
\begin{aligned}
&\mathbf{E}\left(Z^4\right) - 12\mathbf{E}\left(Z^3\right)\\
&= 12\nu^2 - 48\nu\\
&+\frac{3}{q_1}\sum_{p=1,p\neq q_1/2}^{q_1-1} q_1^2(p-1)!\binom{q_1-1}{p-1}^2 p!\binom{q_1}{p}^2 (2q_1-2p)!\,\|f\widetilde{\otimes}_p f\|^2_{L^2(T^{2(q_1-p)})}\\
&+\frac{3}{2}\frac{(q_1!)^5}{((q_1/2)!)^6}\left\|f\widetilde{\otimes}_{q_1/2}f - c_{q_1}f\right\|^2_{L^2(T^{q_1})}\\
&+\frac{3}{q_2}\sum_{p=1,p\neq q_2/2}^{q_2-1} q_2^2(p-1)!\binom{q_2-1}{p-1}^2 p!\binom{q_2}{p}^2 (2q_2-2p)!\,\|h\widetilde{\otimes}_p h\|^2_{L^2(T^{2(q_2-p)})}\\
&+\frac{3}{2}\frac{(q_2!)^5}{((q_2/2)!)^6}\left\|h\widetilde{\otimes}_{q_2/2}h - c_{q_2}h\right\|^2_{L^2(T^{q_2})}.
\end{aligned}
$$

From (ii), it follows that

$$\begin{aligned}
&\frac{3}{q_1}\sum_{p=1,p\neq q_1/2}^{q_1-1} q_1^2(p-1)!\binom{q_1-1}{p-1}^2 p!\binom{q_1}{p}^2(2q_1-2p)!\,\|f\widetilde{\otimes}_p f\|^2_{L^2(T^{2(q_1-p)})}\\
&+\frac{3}{2}\frac{(q_1!)^5}{((q_1/2)!)^6}\left\|f\widetilde{\otimes}_{q_1/2}f-c_{q_1}f\right\|^2_{L^2(T^{q_1})}\\
&+\frac{3}{q_2}\sum_{p=1,p\neq q_2/2}^{q_2-1} q_2^2(p-1)!\binom{q_2-1}{p-1}^2 p!\binom{q_2}{p}^2(2q_2-2p)!\,\|h\widetilde{\otimes}_p h\|^2_{L^2(T^{2(q_2-p)})}\\
&+\frac{3}{2}\frac{(q_2!)^5}{((q_2/2)!)^6}\left\|h\widetilde{\otimes}_{q_2/2}h-c_{q_2}h\right\|^2_{L^2(T^{q_2})}=0,
\end{aligned}$$

which leads to the conclusion as all the summands are positive, that is

$$\begin{aligned}
&\left\|f\widetilde{\otimes}_{q_1/2}f-c_{q_1}f\right\|_{L^2(T^{q_1})}=\left\|h\widetilde{\otimes}_{q_2/2}h-c_{q_2}h\right\|_{L^2(T^{q_2})}=0 \text{ and}\\
&\|f\widetilde{\otimes}_p f\|_{L^2(T^{2(q_1-p)})}=\|h\widetilde{\otimes}_r h\|_{L^2(T^{2(q_2-p)})}=0
\end{aligned}\tag{3.14}$$

for every $p=1,...,q_1-1$ such that $p\neq q_1/2$ and for every $r=1,...,q_2-1$ such that $r\neq q_2/2$; This implies

$$\begin{aligned}
&\left\|f\widetilde{\otimes}_{q_1/2}f-c_{q_1}f\right\|_{L^2(T^{q_1})}=\left\|h\widetilde{\otimes}_{q_2/2}h-c_{q_2}h\right\|_{L^2(T^{q_2})}=0 \text{ and}\\
&\|f\otimes_p f\|_{L^2(T^{2(q_1-p)})}=\|h\otimes_r h\|_{L^2(T^{2(q_2-p)})}=0
\end{aligned}\tag{3.15}$$

for every $p=1,...,q_1-1$ such that $p\neq q_1/2$ and for every $r=1,...,q_2-1$ such that $r\neq q_2/2$ (see [NP09a], Theorem 1.2.).
We will compute $\mathbf{E}\left((2\nu+2Z-G_Z)^2\right)$. Let us start with G_Z.

$$\begin{aligned}
G_Z &= \left\langle DZ,-DL^{-1}Z\right\rangle_{L^2(T)}=\left\langle DI_{q_1}(f)+DI_{q_2}(h),-DL^{-1}I_{q_1}(f)-DL^{-1}I_{q_2}(h)\right\rangle_{L^2(T)}\\
&= \left\langle DI_{q_1}(f),-DL^{-1}I_{q_1}(f)\right\rangle_{L^2(T)}+\left\langle DI_{q_2}(h),-DL^{-1}I_{q_2}(h)\right\rangle_{L^2(T)}\\
&+ \left\langle DI_{q_1}(f),-DL^{-1}I_{q_2}(h)\right\rangle_{L^2(T)}+\left\langle DI_{q_2}(h),-DL^{-1}I_{q_1}(f)\right\rangle_{L^2(T)}.
\end{aligned}$$

From Lemma 10, it follows that

$$\left\langle DI_{q_1}(f),-DL^{-1}I_{q_2}(h)\right\rangle_{L^2(T)}=\left\langle DI_{q_2}(h),-DL^{-1}I_{q_1}(f)\right\rangle_{L^2(T)}=0.$$

Thus,

$$G_Z=q_1^{-1}\|DI_{q_1}(f)\|^2_{L^2(T)}+q_2^{-1}\|DI_{q_2}(h)\|^2_{L^2(T)}.$$

It follows that

$$\begin{aligned}
&\mathbf{E}\left((2\nu+2Z-G_Z)^2\right)\\
&=\mathbf{E}\left(\left(2\nu_1+2\nu_2+2I_{q_1}(f)+2I_{q_2}(h)-q_1^{-1}\|DI_{q_1}(f)\|^2_{L^2(T)}-q_2^{-1}\|DI_{q_2}(h)\|^2_{L^2(T)}\right)^2\right)\\
&=\mathbf{E}\left(\left(q_1^{-1}\|DI_{q_1}(f)\|^2_{L^2(T)}-2I_{q_1}(f)-2\nu_1\right)^2\right)\\
&+\mathbf{E}\left(\left(q_2^{-1}\|DI_{q_2}(h)\|^2_{L^2(T)}-2I_{q_2}(h)-2\nu_2\right)^2\right)\\
&+2\mathbf{E}\left(\left(q_1^{-1}\|DI_{q_1}(f)\|^2_{L^2(T)}-2I_{q_1}(f)-2\nu_1\right)\left(q_2^{-1}\|DI_{q_2}(h)\|^2_{L^2(T)}-2I_{q_2}(h)-2\nu_2\right)\right).
\end{aligned}$$

We use Lemma 9 to write

$$\mathbf{E}\left(\left(q_1^{-1}\left\|DI_{q_1}(f)\right\|^2_{L^2(T)} - 2I_{q_1}(f) - 2\nu_1\right)\left(q_2^{-1}\left\|DI_{q_2}(h)\right\|^2_{L^2(T)} - 2I_{q_2}(h) - 2\nu_2\right)\right) = 0.$$

Thus,

$$\begin{aligned}\mathbf{E}\left((2\nu + 2Z - G_Z)^2\right) &= q_1^{-1}\mathbf{E}\left(\left(\|DI_{q_1}(f)\|^2_{L^2(T)} - 2q_1 I_{q_1}(f) - 2q_1\nu_1\right)^2\right)\\ &+ q_2^{-1}\mathbf{E}\left(\left(\|DI_{q_2}(h)\|^2_{L^2(T)} - 2q_2 I_{q_2}(h) - 2q_2\nu_2\right)^2\right).\end{aligned}$$

Relation (3.15) and the calculations contained in [NP09a] imply that the above two summands vanish.
It finally follows from this that

$$\mathbf{E}\left((2\nu + 2Z - G_Z)^2\right) = 0.$$

Proof of (i) → (ii). Suppose that *(ii)* holds. We have proven that

$$\mathbf{E}\left((2\nu + 2Z - G_Z)^2\right) = 0 \Rightarrow \begin{cases} \mathbf{E}\left(\left(\|DI_{q_1}(f)\|^2_{L^2(T)} - 2q_1 I_{q_1}(f) - 2q_1\nu_1\right)^2\right) = 0 \\ \mathbf{E}\left(\left(\|DI_{q_2}(h)\|^2_{L^2(T)} - 2q_2 I_{q_2}(h) - 2q_2\nu_2\right)^2\right) = 0. \end{cases}$$

From Theorem 1.2 in [NP09a] it follows that $I_{q_1}(f) \sim F(\nu_1)$ and $I_{q_2}(h) \sim F(\nu_2)$. $I_{q_1}(f)$ and $I_{q_2}(h)$ being independent, we use the convolution property of Gamma random variables to state that $Z = I_{q_1}(f) + I_{q_2}(h) \sim F(\nu_1 + \nu_2) \sim F(\nu)$. ■

Remark 5. *The proof of the above theorem shows that the affirmations (i) and (ii) are equivalent with relations (3.8), (3.9), (3.14) and (3.15).*

Following exactly the lines of the proof of Theorem 9 it is possible to characterize random variables given by a sum of independent multiple integrals that converges in law to a Gamma distribution.

Theorem 10. *Fix $\nu_1, \nu_2, \nu > 0$ such that $\nu_1 + \nu_2 = \nu$ and let $F(\nu)$ be a real-valued random variable with characteristic function given by (3.1). Fix two even integers $q_1 \geq 2$ and $q_2 \geq 2$. For any sequence $(f_k)_{k\geq 1} \subset L^2(T^{q_1})$ and $(h_k)_{k\geq 1} \subset L^2(T^{q_2})$ (f_k and h_k are symmetric for every $k \geq 1$) such that*

$$\mathbf{E}\left(I_{q_1}(f_k)^2\right) = q_1!\,\|f_k\|^2_{L^2(T^{q_1})} \underset{k\to+\infty}{\longrightarrow} 2\nu_1 \quad \text{and} \quad \mathbf{E}\left(I_{q_2}(h_k)^2\right) = q_2!\,\|h_k\|^2_{L^2(T^{q_2})} \underset{k\to+\infty}{\longrightarrow} 2\nu_2,$$

and such that $X_k = I_{q_1}(f_k)$ and $Y_k = I_{q_2}(h_k)$ are independent for any $k \geq 1$, define the random variable

$$Z_k = X_k + Y_k = I_{q_1}(f_k) + I_{q_2}(h_k) \quad \forall k \geq 1.$$

Under those conditions, the following two conditions are equivalent :

(i) $\mathbf{E}\left(\left(2\nu + 2Z_k - \langle DZ_k, -DL^{-1}Z_k\rangle_{L^2(T)}\right)^2\right) \underset{k\to+\infty}{\longrightarrow} 0$ *;*

(ii) $Z_k \underset{k\to+\infty}{\overset{\text{Law}}{\longrightarrow}} F(\nu)$ *;*

Cramér's theorem for Gamma random variables in the setting of multiple stochastic integrals is a corollary of Theorem 9. We have the following :

Theorem 11. *Let $Z = X + Y = I_{q_1}(f) + I_{q_2}(h)$, $q_1, q_2 \geq 2$, $f \in L^2(T^{q_1}), h \in L^2(T^{q_2})$ symmetric, be such that X, Y are independent and*

$$\mathbf{E}\left(Z^2\right) = 2\nu, \mathbf{E}\left(X^2\right) = q_1! \left\|f\right\|^2_{L^2(T^{q_1})} = 2\nu_1, \mathbf{E}\left(Y^2\right) = q_2! \left\|h\right\|^2_{L^2(T^{q_2})} = 2\nu_2$$

with $\nu = \nu_1 + \nu_2$. Furthermore, let's assume that $Z \sim F(\nu)$. Then,

$$X \sim F(\nu_1) \quad \text{and} \quad Y \sim F(\nu_2).$$

Proof : Theorem 9 states that $Z \sim F(\nu) \Leftrightarrow \mathbf{E}\left((2\nu + 2Z - G_Z)^2\right) = 0$ and we proved that

$$\mathbf{E}\left((2\nu + 2Z - G_Z)^2\right) = \mathbf{E}\left((2\nu_1 + 2X - G_X)^2\right) + \mathbf{E}\left((2\nu_2 + 2Y - G_Y)^2\right).$$

Both summands being positive, it follows that

$$\mathbf{E}\left((2\nu_1 + 2X - G_X)^2\right) = 0 \quad \text{and} \quad \mathbf{E}\left((2\nu_2 + 2Y - G_Y)^2\right) = 0.$$

Applying theorem 9 to X and Y separately gives us $\mathbf{E}\left((2\nu_1 + 2X - G_X)^2\right) \Leftrightarrow X \sim F(\nu_1)$ and $\mathbf{E}\left((2\nu_2 + 2Y - G_Y)^2\right) \Leftrightarrow Y \sim F(\nu_2)$. ■

It is immediate to give an asymptotic version of Theorem 11.

Theorem 12. *Let $Z_k = X_k + Y_k = I_{q_1}(f_k) + I_{q_2}(h_k)$, $f_k \in L^2(T^{q_1}), h_k \in L^2(T^{q_2})$ symmetric for $k \geq 1$, $q_1, q_2 \geq 2$, be such that X_k, Y_k are independent for every $k \geq 1$ and*

$$\mathbf{E}\left(Z_k^2\right) \underset{k \to +\infty}{\longrightarrow} 2\nu, \mathbf{E}\left(X_k^2\right) = q_1! \left\|f\right\|^2_{L^2(T^{q_1})} \underset{k \to +\infty}{\longrightarrow} 2\nu_1, \mathbf{E}\left(Y_k^2\right) = q_2! \left\|h\right\|^2_{L^2(T^{q_2})} \underset{k \to +\infty}{\longrightarrow} 2\nu_2$$

with $\nu = \nu_1 + \nu_2$. Furthermore, let's assume that $Z_k \underset{k \to +\infty}{\longrightarrow} F(\nu)$ in distribution. Then,

$$X_k \underset{k \to +\infty}{\longrightarrow} F(\nu_1) \quad \text{and} \quad Y_k \underset{k \to +\infty}{\longrightarrow} F(\nu_2).$$

Remark 6. *i) From Corollary 4.4. in [NP09a] it follows that actually there are no Gamma distributed random variables in a chaos of order bigger or equal than 4. (We actually conjecture that a Gamma distributed random variable given by a multiple integral can only live in the second Wiener chaos). In this sense Theorem 11 contains a limited number of examples. By contrary, the asymptotic Cramér theorem (Theorem 12) is more interesting and more general since there exists a large class of variables which are asymptotically Gamma distributed.*
ii) Theorem 11 cannot be applied directly to random variables with law $\Gamma(a, \lambda)$ (as defined in the Introduction) because such random variables are not centered and then they cannot live in a finite Wiener chaos. But, it is not difficult to understand that if $X = I_{q_1} + c$ is a random variable which is independent of $Y = I_{q_2} + d$ (and assume that the first two moments of X and Y are the same as the moment of the corresponding Gamma distributions), and if $X + Y \sim \Gamma(a + b, \lambda)$ then X has the distribution $\Gamma(a, \lambda)$ and Y has the distribution $\Gamma(b, \lambda)$.
iii) Several results of the paper (Lemmas 1 and 2) holds for strongly independent random variables. Nevertheless, the key results (Theorems 9 and 10 that allows to prove Cramér's theorem and its asymptotic variant are not true for strongly independent random variables (actually the implication ii) $\to$ i) in these results, whose proof is based on the differential equation satisfied by the characteristic function of the Gamma distribution, does not work.

3.4 Counterexample in the general case

We will see in this section that Theorem 11 does not hold for random variables which have a chaos decomposition into an infinite sum of multiple stochastic integrals. We construct a counterexample in this sense. What is more interesting is that the random variables defined in the below example are not only independent, they are *strongly independent* (see the definition above).

Example 1. *Let $\epsilon(\lambda)$ denote the exponential distribution with parameter λ and let $b(p)$ denote the Bernoulli distribution with parameter p. Let $X = A - 1$ and $Y = 2\varpi B - 1$, where $A \sim \epsilon(1)$, $B \sim \epsilon(1)$, $\varpi \sim b(\frac{1}{2})$ and A, B and ϖ are mutually independent. This implies that X and Y are independent. We have $\mathbf{E}(X) = \mathbf{E}(Y) = 0$ as well as $\mathbf{E}(X^2) = 1$ and $\mathbf{E}(Y^2) = 3$. Consider also $Z = X+Y$. Observe that X,Y and Z match every condition of theorem 11, but X and Y are not multiple stochastic integrals in a fixed Wiener chaos (see the next proposition for more details). We have the following : $Z \sim F(2)$, but Y is not Gamma distributed.*

Proof : We know that

$$\mathbf{E}\left(e^{itX}\right) = \mathbf{E}\left(e^{it(A-1)}\right) = e^{-it}\mathbf{E}\left(e^{itA}\right) = \frac{e^{-it}}{1-it}$$

and that

$$\begin{aligned}\mathbf{E}\left(e^{itY}\right) &= \mathbf{E}\left(e^{it(2\varpi B-1)}\right) = e^{-it}\mathbf{E}\left(e^{it2\varpi B}\right) = e^{-it}\left(\frac{1}{2}\mathbf{E}\left(e^{it2B}\right) + \frac{1}{2}\right) \\ &= e^{-it}\left(\frac{1}{2}\frac{1}{1-2it} + \frac{1}{2}\right) = e^{-it}\frac{1-it}{1-2it}.\end{aligned}$$

Observe at this point that the characteristic function of Y proves that Y is not Gamma distributed. Let us compute the characteristic function of Z. We have

$$\mathbf{E}\left(e^{itZ}\right) = \mathbf{E}\left(e^{it(X+Y)}\right) = \mathbf{E}\left(e^{itX}\right)\mathbf{E}\left(e^{itY}\right) = \frac{e^{-it}}{1-it}e^{-it}\frac{1-it}{1-2it} = \frac{e^{-2it}}{1-2it} = \mathbf{E}\left(e^{itF(2)}\right).$$

■

Remark 7. *It is also possible to construct a similar example for the laws $\Gamma(a,\lambda), \Gamma(b,\lambda)$ instead of $F(\nu_1), F(\nu_2)$.*

The following proposition shows that this counterexample accounts for independent random variables but also for strongly independent random variables.

Proposition 10. *X and Y as defined in Example 1 are strongly independent.*

Proof : In order to prove that X and Y are strongly independent, we need to compute their Wiener chaos expansions in order to emphasize the fact that all the components of these Wiener Chaos expansions are mutually independent. Consider a standard Brownian motion B indexed on $L^2(T) = L^2((0,T))$. Consider $h_1, ..., h_5 \in L^2(T)$ such that $\|h_i\|_{L^2(T)} = 1$ for every $1 \leq i \leq 5$ and such that $W(h_i)$ and $W(h_j)$ are independent for every $1 \leq i, j \leq 5, i \neq j$. First notice that the random variables $A = \frac{1}{2}\left(W(h_1)^2 + W(h_2)^2\right)$ and $B = \frac{1}{2}\left(W(h_4)^2 + W(h_5)^2\right)$ are independent (this is obvious) and have the exponential distribution with parameter 1. Also, note that the random variable $\varpi = \frac{1}{2}\mathrm{sign}(W(h_3)) + \frac{1}{2}$

has the Bernoulli distribution and is independent from A and B. As in Example 1, set $X = A - 1$ and $Y = 2\varpi B - 1$. X and Y are as defined in Example 1. Let us now compute their Wiener chaos decompositions. We have

$$A = \frac{1}{2}\left(W(h_1)^2 + W(h_2)^2\right) = \frac{1}{2}\left(I_1(h_1)^2 + I_1(h_2)^2\right) = \frac{1}{2}\left(2 + I_2(h_1^{\otimes 2}) + I_2(h_2^{\otimes 2})\right),$$

and similarly $B = \frac{1}{2}\left(2 + I_2(h_4^{\otimes 2}) + I_2(h_5^{\otimes 2})\right)$. Therefore, we have

$$X = I_2\left(\frac{h_1^{\otimes 2} + h_2^{\otimes 2}}{2}\right).$$

From [HN05], Lemma 3, we know that

$$\operatorname{sign}(W(h_3)) = \sum_{k\geq 0} b_{2k+1} I_{2k+1}(h_3^{\otimes(2k+1)}),$$

where $b_{2k+1} = \frac{2(-1)^k}{(2k+1)\sqrt{2\pi}k!2^k}$. It follows that $\varpi = \frac{1}{2} + \frac{1}{2}\sum_{k\geq 0} b_{2k+1} I_{2k+1}(h_3^{\otimes(2k+1)})$, and

$$\begin{aligned}
Y &= (1 + \sum_{k\geq 0} b_{2k+1} I_{2k+1}(h_3^{\otimes(2k+1)}))(1 + \frac{1}{2} I_2(h_4^{\otimes 2}) + \frac{1}{2} I_2(h_5^{\otimes 2})) - 1 \\
&= \frac{1}{2} I_2(h_4^{\otimes 2}) + \frac{1}{2} I_2(h_5^{\otimes 2}) + \sum_{k\geq 0} b_{2k+1} I_{2k+1}(h_3^{\otimes(2k+1)}) + \frac{1}{2}\sum_{k\geq 0} b_{2k+1} I_{2k+1}(h_3^{\otimes(2k+1)}) I_2(h_4^{\otimes 2}) \\
&\quad + \frac{1}{2}\sum_{k\geq 0} b_{2k+1} I_{2k+1}(h_3^{\otimes(2k+1)}) I_2(h_5^{\otimes 2}).
\end{aligned}$$

Using the multiplication formula for multiple stochastic integrals, we obtain

$$\begin{aligned}
Y &= \frac{1}{2} I_2(h_4^{\otimes 2}) + \frac{1}{2} I_2(h_5^{\otimes 2}) + \sum_{k\geq 0} b_{2k+1} I_{2k+1}(h_3^{\otimes(2k+1)}) \\
&\quad + \frac{1}{2}\sum_{k\geq 0} b_{2k+1} \sum_{r=0}^{(2k+1)\wedge 2} r! \binom{2}{r}\binom{2k+1}{r} I_{2k+3-2r}(h_3^{\otimes(2k+1)} \otimes_r h_4^{\otimes 2}) \\
&\quad + \frac{1}{2}\sum_{k\geq 0} b_{2k+1} \sum_{r=0}^{(2k+1)\wedge 2} r! \binom{2}{r}\binom{2k+1}{r} I_{2k+3-2r}(h_3^{\otimes(2k+1)} \otimes_r h_5^{\otimes 2}).
\end{aligned}$$

At this point, it is clear that X and Y are strongly independent. ■

Chapitre 4

Malliavin Calculus and Self Normalized Sums

Solesne BOURGUIN * and Ciprian A. TUDOR † ‡
Université Paris 1 and *Université Lille 1*

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Abstract

We study the self-normalized sums of independent random variables from the perspective of the Malliavin calculus. We give the chaotic expansion for them and we prove a Berry-Esséen bound with respect to several distances.

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4.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t\geq 0}$ a Brownian motion on this space. Let F be a random variable defined on Ω which is differentiable in the sense of the Malliavin calculus. Then using the so-called Stein's method introduced by Nourdin and Peccati in [NP09c] (see also [NP09b] and [NP10]), it is possible to measure the distance between the law of F and the standard normal law $\mathcal{N}(0,1)$. This distance can be defined in several ways, such as the Kolmogorov distance, the Wasserstein distance, the total variation distance or the Fortet-Mourier distance. More precisely we have, if $\mathcal{L}(F)$ denotes the law of F,

$$d(\mathcal{L}(F), \mathcal{N}(0,1)) \leq c\sqrt{\mathbf{E}\left(1 - \langle DF, D(-L)^{-1}F\rangle_{L^2([0,1])}\right)^2}.$$

∗. SAMM, Université de Paris 1 Panthéon-Sorbonne, 90, rue de Tolbiac, 75634, Paris, France. Email : `solesne.bourguin@univ-paris1.fr`

†. Laboratoire Paul Painlevé, Université de Lille 1, F-59655 Villeneuve d'Ascq, France. Email : `tudor@math.univ-lille1.fr`

‡. Associate member of the team Samm, Université de Paris 1 Panthéon-Sorbonne.

Here D denotes the Malliavin derivative with respect to W, and L is the generator of the Ornstein-Uhlenbeck semigroup. We will explain in the next section how these operators are defined. The constant c is equal to 1 in the case of the Kolmogorov distance as well as in the case of the Wasserstein distance, $c = 2$ for the total variation distance and $c = 4$ in the case of the Fortet-Mourier distance.

Our purpose is to apply these techniques to self-normalized sums. Let us recall some basic facts on this topic. We refer to [dlPLS09] and the references therein for a more detailed exposition. Let $X_1, X_2, \cdots$ be independent random variables. Set $S_n = \sum_{i=1}^n X_i$ and $V_n^2 = \sum_{i=1}^n X_i^2$. Then $\frac{S_n}{V_n}$ converges in distribution as $n \to \infty$ to the standard normal law $\mathcal{N}(0,1)$ if and only if $\mathbf{E}(X) = 0$ and X is in the domain of attraction of the standard normal law (see [dlPLS09], Theorem 4.1). The "if" part of the theorem has been known for a long time (it appears in [Mal81]) while the "only if" part remained open until its proof in [GGM97]. The Berry-Esséen theorem for self-normalized sums has been also widely studied. We refer to [BG96] and [Sha05] (see also [BBG96], [BGT97] for the situation where the random variables X_i are non i.i.d.). These results say that the Kolmogorov distance between the law of $\frac{S_n}{V_n}$ and the standard normal law is less than

$$C\left(B_n^{-2}\sum_{i=1}^N \mathbf{E}\left(X_i^2 1_{(|X_i|>B_n)}\right) + B_n^{-3}\sum_{i=1}^N \mathbf{E}\left(X_i^3 1_{(|X_i|\geq B_n)}\right)\right)$$

where $B_n = \sum_{i=1}^n \mathbf{E}(X_i^2)$ and C is an absolute constant. We mention that, as far as we know, these results only exist for the Kolmogorov distance. To use our techniques based on the Malliavin calculus and multiple stochastic integrals, we will put ourselves on a Gaussian space where we will consider the following particular case : the random variables X_i are the increments of the Wiener process $X_i = W_i - W_{i-1}$. The Berry-Esséen bound from above reduces to (see [dlPLS09], page 53) : for $2 < p \leq 3$

$$\sup_{z\in\mathbb{R}} |\mathbf{P}(F_n \leq z) - \Phi(z)| \leq 25\mathbf{E}\left(|Z|^p\right) n^{1-\frac{p}{2}} \tag{4.1}$$

where Z is a standard normal random variable and Φ is its repartition function. In particular for $p = 3$ we get

$$\sup_{z\in\mathbb{R}} |\mathbf{P}(F_n \leq z) - \Phi(z)| \leq 25\mathbf{E}\left(|Z|^3\right) n^{-\frac{1}{2}}. \tag{4.2}$$

We will compare our result with the above relation (4.2). The basic idea is as follows : we are able to find the chaos expansion into multiple Wiener-Itô integrals of the random variable $\frac{S_n}{V_n}$ for every $n \geq 2$ and to compute its Malliavin derivative. Note that the random variable $\frac{S_n}{V_n}$ has a decomposition into an infinite sum of multiple integrals in contrast to the examples provided in the papers [BT11a], [NP09c], [NP09b]. Then we compute the Berry-Esséen bound given by $\sqrt{\mathbf{E}\left(1 - \langle DF, D(-L)^{-1}F\rangle_{L^2([0,1])}\right)^2}$ by using properties of multiple stochastic integrals. Of course, we cannot expect to obtain a rate of convergence better than $c\frac{1}{\sqrt{n}}$, but we have an explicit expression of the constant appearing in this bound and our method is available for several distances between the laws of random variables (not limited to the Kolmogorov distance). This aspect of the problem seems to be new. This computation of the Berry-Esséen bound is also interesting in and of itself as it brings to light original relations involving Gaussian measure and Hermite polynomials. It gives an exact expression of the chaos expansion of the self normalized sum and it also shows that the convergence to the normal law of $\frac{S_n}{V_n}$ is uniform with respect to the chaos, in the

sense that every chaos of $\frac{S_n}{V_n}$ is convergent to the standard normal law and that the rate is the same for every chaos.
We have organized our paper as follows : Section 2 contains the elements of the Malliavin calculus needed in the paper and in Section 3 we discuss the chaos decomposition of self-normalized sums as well as study the asymptotic behavior of the coefficients appearing in this expansion. Section 4 contains the computation of the Berry-Esséen bound given in terms of the Malliavin calculus.

4.2 Preliminaries

We will begin by describing the basic tools of multiple Wiener-Itô integrals and Malliavin calculus that will be needed in our paper. Let $(W_t)_{t\in[0,T]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, P)$. If $f \in L^2([0,T]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W. We refer to [Nua06] for a detailed exposition of the construction and the properties of multiple Wiener-Itô integrals.
Let $f \in \mathcal{S}_n$, which means that there exists $n \geq 1$ integers such that

$$f := \sum_{i_1,\cdots,i_n} c_{i_1,\cdots,i_n} 1_{A_{i_1}\times\cdots\times A_{i_n}}$$

where the coefficients satisfy $c_{i_1,\cdots,i_n} = 0$ if two indices i_k and i_ℓ are equal and the sets $A_i \in \mathcal{B}([0,T])$ are disjoints. For a such step function f we define

$$I_n(f) := \sum_{i_1,\cdots,i_n} c_{i_1,\cdots,i_n} W(A_{i_1})\cdots W(A_{i_n})$$

where we put $W([a,b]) = W_b - W_a$. It can be seen that the application I_n constructed above from $\mathcal{S}_n$ equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{L^2([0,T]^n)}$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}_n$, i.e. for m, n positive integers,

$$\begin{aligned} \mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle f,g\rangle_{L^2([0,T]^n)} \quad \text{if } m=n, \\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0 \quad \text{if } m\neq n. \end{aligned}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1,\cdots,x_x) = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} f(x_{\sigma(1)},\cdots,x_{\sigma(n)}).$$

Since the set $\mathcal{S}_n$ is dense in $L^2([0,T]^n)$ for every $n \geq 2$, the mapping I_n can be extended to an isometry from $L^2([0,T]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension. Note also that I_n can be viewed as an iterated stochastic integral (this follows e.g. by Itô's formula)

$$I_n(f) = n!\int_0^1\int_0^{t_n}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dW_{t_1}\cdots dW_{t_n}$$

We recall the product for two multiple integrals (see [Nua06]) : if $f \in L^2([0,T]^n)$ and $g \in L^2([0,T]^m)$ are symmetric, then it holds that

$$I_n(f)I_m(g) = \sum_{\ell=0}^{m\wedge n} \ell! C_m^\ell C_n^\ell I_{m+n-2\ell}(f\otimes_\ell g) \tag{4.3}$$

where the contraction $f \otimes_\ell g$ belongs to $L^2([0,T]^{m+n-2\ell})$ for $\ell = 0, 1, \cdots, m \wedge n$ and is given by

$$\begin{aligned} & (f \otimes_\ell g)(s_1, \cdots, s_{n-\ell}, t_1, \cdots, t_{m-\ell}) \\ = & \int_{[0,T]^\ell} f(s_1, \cdots, s_{n-\ell}, u_1, \cdots, u_\ell) g(t_1, \cdots, t_{m-\ell}, u_1, \cdots, u_\ell) du_1 \cdots du_\ell. \end{aligned}$$

We recall that any square integrable random variable that is measurable with respect to the σ-algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n \geq 0} I_n(f_n) \tag{4.4}$$

where $f_n \in L^2([0,1]^n)$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}(F)$. Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n \geq 0} n I_n(f_n) \text{ and } L^{-1}F = -\sum_{n \geq 1} \frac{1}{n} I_n(f_n)$$

if F is given by (4.4). We denote by D the Malliavin derivative operator that acts on smooth functionals of the form $F = g(W(\varphi_1), \cdots, W(\varphi_n))$ where g is a smooth function with compact support and $\varphi_i \in L^2([0,1])$. For $i = 1, \cdots, n$, the derivative operator is defined by

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(B(\varphi_1), \cdots, B(\varphi_n))\varphi_i.$$

The operator D can be extended to the closure $\mathbb{D}^{p,2}$ of smooth functionals with respect to the norm

$$\|F\|_{p,2}^2 = \mathbf{E}\left(F^2\right) + \sum_{i=1}^p \mathbf{E}\left(\|D^i F\|_{L^2([0,1]^i)}^2\right)$$

where the i^{th} order Malliavin derivative D^i is defined iteratively.
Let us recall how this derivative acts for random variables in a finite chaos. If $f \in L^2([0,T]^n)$ is a symmetric function, we will use the following rule to differentiate in the Malliavin sense

$$D_t I_n(f) = n\, I_{n-1}(f(\cdot, t)), \qquad t \in \mathbb{R}.$$

Let us also recall how the distances between the laws of random variables are defined. We have

$$d(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{h \in \mathcal{A}} \left(|\mathbf{E}(h(X)) - \mathbf{E}(h(Y))|\right)$$

where $\mathcal{A}$ denotes a set of functions. When $\mathcal{A} = \{h : \|h\|_L \geq 1\}$ (here $\|\cdot\|_L$ is the Lipschitz norm) we obtain the Wasserstein distance, when $\mathcal{A} = \{h : \|h\|_{BL} \geq 1\}$ (with $\|\cdot\|_{LB} = \|\cdot\|_L + \|\cdot\|_\infty$) we get the Fortet-Mourier distance, when $\mathcal{A}$ is the set of indicator functions of Borel sets we obtain the total variation distance, and when $\mathcal{A}$ is the set of indicator functions of the form $1_{(-\infty,z)}$ with $z \in \mathbb{R}$, we obtain the Kolmogorov distance that has been presented above.

4.3 Chaos decomposition of self-normalized sums

The tools of the Malliavin calculus presented above can be successfully applied in order to study self-normalized sums. Because of the nature of Malliavin calculus, we put ourselves in a Gaussian setting and we consider $X_i = W_i - W_{i-1}$ to be the increments of a classical Wiener process W. We then consider the sums

$$S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad V_n^2 = \sum_{i=1}^{n} X_i^2$$

as well as the *self-normalized sum* F_n defined by

$$F_n = \frac{S_n}{V_n} = \frac{W_n}{\left(\sum_{i=1}^{n}(W_{i+1} - W_i)^2\right)^{\frac{1}{2}}}. \tag{4.5}$$

Let us now concentrate our efforts on finding the chaotic decomposition of the random variable F_n. This will be the key to computing Berry-Esséen bounds for the distance between the law of F_n and the standard normal law in the next section.

Lemma 11. *Let F_n be given by (4.5) and let $f : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$f(x_1, \cdots, x_n) = \frac{x_1 + \cdots + x_n}{(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}}. \tag{4.6}$$

Let $\varphi_i = 1_{[i-1,i]}$ for $i = 1, \cdots, n$. Then for every $n \geq 2$, we have

$$F_n = \sum_{k \geq 0} \frac{1}{k!} \sum_{i_1, \cdots, i_k = 1}^{n} a_{i_1, \cdots, i_k} I_k \left(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}\right)$$

with

$$a_{i_1, \cdots, i_k} \overset{def}{=} \mathbf{E}\left(\frac{\partial^k f}{\partial x_{i_1}, \cdots, x_{i_k}}\left(W(\varphi_1), \cdots, W(\varphi_n)\right)\right). \tag{4.7}$$

Proof : First note that F_n can be written as

$$F_n = f\left(W(\varphi_1), \cdots, W(\varphi_n)\right).$$

We can also write

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} f_i(x_1, \cdots, x_n),$$

where $f_i(x_1, \cdots, x_n)$ is defined by

$$f_i(x_1, \cdots, x_n) = \frac{x_i}{(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}}.$$

The chaotic decomposition of $f_i\left(W(\varphi_1), \cdots, W(\varphi_n)\right)$ was obtained (in a slightly different setting) by Hu and Nualart in the proof of Proposition 10 in [HN05]. They proved that

$$f_i\left(W(\varphi_1), \cdots, W(\varphi_n)\right) = \sum_{k=0}^{\infty} \sum_{j_1, \cdots, j_k = 1}^{n} b_{i, j_1, \cdots, j_k} I_k\left(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}\right),$$

where

$$b_{i, j_1, \cdots, j_k} = \frac{(-1)^k}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} \left[\frac{\partial^k}{\partial x_{j_1} \cdots \partial x_{j_k}} e^{-\frac{(x_1 + \cdots + x_n)^2}{2}}\right] f_i(x_1, \cdots, x_n) dx_1 \cdots dx_n.$$

Define $b_{j_1,\cdots,j_k} = \sum_{i=1}^n b_{i,j_1,\cdots,j_k}$. By the above result, we have

$$b_{j_1,\cdots,j_k} = \frac{(-1)^k}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} \left[\frac{\partial^k}{\partial x_{j_1} \cdots \partial x_{j_k}} e^{-\frac{(x_1+\cdots+x_n)^2}{2}} \right] f(x_1, \cdots, x_n) dx_1 \cdots dx_n.$$

Thus,

$$F_n = \sum_{k\geq 0} \sum_{i_1,\cdots,i_k=1}^n b_{i_1,\cdots,i_k} I_k \left(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \right).$$

Finally, using the Gaussian integration by part formula yields $b_{i_1,\cdots,i_k} = a_{i_1,\cdots,i_k}$, where $a_{i_1,\cdots,i_k}$ is defined by (4.7), which concludes the proof. ■

Remark 8. *The coefficients $a_{i_1,\cdots,i_k}$ also depend on n. We omit n in their notation in order to simplify the presentation.*

4.3.1 Computing the coefficients in the chaos expansion

In this subsection, we explicitly compute the coefficients $a_{i_1,\cdots,i_k}$ appearing in Lemma 11. Let $\mathbf{H}_n(x)$ denote the n^{th} Hermite polynomial :

$$\mathbf{H}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

Define

$$\begin{aligned} W_n &\stackrel{\text{def}}{=} W(\varphi_1) + W(\varphi_2) + \cdots + W(\varphi_n) \\ V_n &\stackrel{\text{def}}{=} \Big(\sum_{i=1}^n W(\varphi_i)^2 \Big)^{1/2} \end{aligned}$$

Let us first give the following lemma that can be proved using integration by parts.

Lemma 12. *For every $1 \leq i_1, ., i_k \leq n$, let $a_{i_1,\cdots i_k}$ be as defined in (4.7). Let $d_r, 1 \leq r \leq n$ denote the number of times the integer r appears in the sequence $\{i_1, i_2, \cdots, i_k\}$ with $\sum_{r=1}^n d_r = k$. Then we have*

$$a_{i_1,\cdots i_k} = \mathbf{E} \left(\frac{W_n}{V_n} \prod_{r=1}^n \mathbf{H}_{d_r}(W(\varphi_r)) \right).$$

Proof : If $X \sim \mathcal{N}(0,1)$, then for any $g \in C^{(n)}(\mathbb{R})$ with g and its derivatives having polynomial growth at infinity, we have the Gaussian integration by parts formula

$$\mathbf{E}(g^{(n)}(X)) = \mathbf{E}(g(X)\mathbf{H}_n(X)).$$

where $g^{(n)}(x) \stackrel{\text{def}}{=} \frac{d^n}{dx^n} g(x)$.
Notice that the function f defined in (4.6) satisfies $|f(x)| \leq C|x|, \forall x \in \mathbb{R}^n$ for a constant C, and thus applying the above integration by parts formula recursively yields

$$\begin{aligned} a_{i_1,\cdots i_k} &= \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \left(\frac{\partial^k f}{\partial x_1^{d_1}, \cdots, x_n^{d_n}} \right) (x_1, \cdots, x_n)\, e^{-\frac{x_1^2}{2}} \cdots e^{-\frac{x_n^2}{2}} dx_1 \cdots dx_n \\ &= \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \left(\frac{\partial^k f}{\partial x_1^{d_1}, \cdots, x_{n-1}^{d_{n-1}}} \right) (x_1, \cdots, x_n)\; \mathbf{H}_{d_n}(x_n)\, e^{-\frac{x_1^2}{2}} \cdots e^{-\frac{x_n^2}{2}} dx_1 \cdots dx_n \\ &= \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(x_1, \cdots, x_n) \prod_{r=1}^n \mathbf{H}_{d_r}(x_r)\, e^{-\frac{x_1^2}{2}} \cdots e^{-\frac{x_n^2}{2}} dx_1 \cdots dx_n \\ &= \mathbf{E} \left(\frac{W_n}{V_n} \prod_{r=1}^n \mathbf{H}_{d_r}(W(\varphi_r)) \right). \end{aligned}$$

This concludes the proof of the Lemma. ■

The next step in the calculation of the coefficient is to notice that $a_{i_1,\cdots i_k} = 0$ when k is even. This is the object of the following Lemma.

Lemma 13. *If k is even, then*

$$a_{i_1,\cdots i_k} = 0.$$

Proof : Let k be an even number and $d_1, d_2, \cdots, d_n$ be as defined in Lemma 12. By Lemma 12, we have

$$a_{i_1,\cdots i_k} = \sum_{u=1}^{n} \mathbf{E}\left(\frac{W(\varphi_u)}{V_n}\prod_{r=1}^{n}\mathbf{H}_{d_r}\left(W(\varphi_r)\right)\right). \tag{4.8}$$

Note that the product $\prod_{r=1}^{n}\mathbf{H}_{d_r}\left(W(\varphi_r)\right)$ is an even function of $(W(\varphi_1), W(\varphi_2), \cdots, W(\varphi_n))$. Indeed, since k is even and $\sum_{r=1}^{n} d_r = k$, either all of the integers $d_r, r \le n$ are even or there is an even number of odd integers in $d_r, r \le n$. In either case the product $\prod_{r=1}^{n}\mathbf{H}_{d_r}\left(W(\varphi_r)\right)$ is an even function of $(W(\varphi_1), W(\varphi_2), \cdots, W(\varphi_n))$, since $\mathbf{H}_m(x) = \mathbf{H}_m(-x)$ for all even $m \in \mathbb{N}$ and $\mathbf{H}_m(x) = -\mathbf{H}_m(-x)$ for all odd $m \in \mathbb{N}$.
Thus for each $u \le n$, the expression $\frac{W(\varphi_u)}{V_n}\prod_{r=1}^{n}\mathbf{H}_{d_r}\left(W(\varphi_r)\right)$ is an odd function of $W(\varphi_u)$ and thus has expectation zero since $W(\phi_u)$ is a standard Gaussian random variable. The fact that (4.8) is a sum of such expectations concludes the proof. ■

As a consequence of Lemma 13, we have

$$F_n = \sum_{k\ge 0}\frac{1}{(2k+1)!}\sum_{i_1,\cdots,i_{2k+1}=1}^{n} a_{i_1,\cdots,i_{2k+1}} I_{2k+1}\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_{2k+1}}\right). \tag{4.9}$$

This implies that in order to compute the coefficients $a_{i_1,\cdots i_k}$, it suffices to focus on the case where k is odd. Before stating the first result in this direction, let us give the following technical lemma.

Lemma 14. *Let $k \ge 0$ be a positive integer and let $d_r, 1 \le r \le n$ denote the number of times the integer r appears in the sequence $\{i_1, i_2, \cdots, i_{2k+1}\}$ with $\sum_{r=1}^{n} d_r = 2k+1$. Then, if there is more than one odd integer in the sequence $d_r, 1 \le r \le n$, for each $1 \le i \le n$,*

$$\mathbf{E}\left[\frac{1}{V_n}W(\varphi_i)\mathbf{H}_{d_1}\left(W(\varphi_1)\right)\mathbf{H}_{d_2}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n}\left(W(\varphi_n)\right)\right] = 0.$$

Proof : Note that the equality $\sum_{r=1}^{n} d_r = 2k+1$ implies that there can only be an odd number of odd integers in the sequence d_r, otherwise the sum $\sum_{r=1}^{n} d_r$ could not be odd. Therefore, more than one odd integer in the sequence d_r means that there are at least three of them. We will prove the Lemma for this particular case of three odd integers in the sequence d_r for the sake of readability of the proof, as the other cases follow with the exact same arguments. Hence, assume that there are three odd integers d_i, d_k and d_l in the sequence $d_r, 1 \le r \le n$. We will first consider the case where i is different than j, k, l.

Then,

$$
\begin{aligned}
&\mathbf{E}\left[\frac{1}{V_n}W(\varphi_i)\mathbf{H}_{d_1}\left(W(\varphi_1)\right)\mathbf{H}_{d_2}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n}\left(W(\varphi_n)\right)\right]\\
&=\frac{1}{(2n)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{x_i\mathbf{H}_{d_1}\left(x_1\right)\cdots\mathbf{H}_{d_n}\left(x_n\right)}{\sqrt{x_1^2+\cdots+x_n^2}}e^{-\frac{1}{2}(x_1^2+\cdots+x_n^2)}dx_1\cdots dx_n\\
&=\frac{1}{(2n)^{\frac{n}{2}}}\int_{\mathbb{R}^{n-1}}x_i\mathbf{H}_{d_1}\left(x_1\right)\cdots\mathbf{H}_{d_{j-1}}\left(x_{j-1}\right)\mathbf{H}_{d_{j+1}}\left(x_{j+1}\right)\cdots\mathbf{H}_{d_n}\left(x_n\right)\\
&\times\left(\int_{\mathbb{R}}\frac{\mathbf{H}_{d_j}\left(x_j\right)}{\sqrt{x_1^2+\cdots+x_n^2}}e^{-\frac{x_j^2}{2}}dx_j\right)\exp\left[-\frac{1}{2}\sum_{\substack{p=1\\p\neq j}}^{n}x_p^2\right]dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n.
\end{aligned}
$$

d_j beeing odd, $\mathbf{H}_{d_j}$ is an odd function of x_j and $x_j\mapsto\frac{\mathbf{H}_{d_j}(x_j)}{\sqrt{x_1^2+\cdots+x_n^2}}e^{-\frac{x_j^2}{2}}$ is also an odd function of x_j. Thus, $\int_{\mathbb{R}}\frac{\mathbf{H}_{d_j}(x_j)}{\sqrt{x_1^2+\cdots+x_n^2}}e^{-\frac{x_j^2}{2}}dx_j=0$ and finally

$$\mathbf{E}\left[\frac{1}{V_n}W(\varphi_i)\mathbf{H}_{d_1}\left(W(\varphi_1)\right)\mathbf{H}_{d_2}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n}\left(W(\varphi_n)\right)\right]=0.$$

The other cases one could encounter is when $i=j$ or $i=k$ or $i=l$ and the proof follows based on the exact same argument. ■

We can now state the following key result that will allow us to perform further calculations in order to explicitly determine the coefficients $a_{i_1,\cdots i_k}$.

Lemma 15. *For every $k\geq 0$ and for every $1\leq i_1,\cdots,i_{2k+1}\leq n$, let $d_r^\star, 1\leq r\leq n$ be the number of times the integer r appears in the sequence $\{i_1,\cdots,i_{2k+1}\}$. Then,*

$$a_{1_1,\cdots,i_{2k+1}}=\mathbf{E}\left[\frac{1}{V_n}W(\varphi_1)\mathbf{H}_{d_1^\star}\left(W(\varphi_1)\right)\mathbf{H}_{d_2^\star}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n^\star}\left(W(\varphi_n)\right)\right]\tag{4.10}$$

if there is only one odd integer in the sequence $d_r^\star, 1\leq r\leq n$. If there is more than one odd integer in the sequence $d_r^\star, 1\leq r\leq n$, we have $a_{1_1,\cdots,i_{2k+1}}=0$.

Remark 9. *Note that in (4.10), it might be understood that $d_1^\star$ is always the only odd integer in $d_r^\star, 1\leq r\leq n$. This is obviously not always the case and if $d_1^\star$ is not the odd integer but let's say, $d_i^\star$ with $1<i\leq n$ is, one can use the equality in law between $W(\varphi_i)$ and $W(\varphi_1)$ to perform an index swap $(i\leftrightarrow 1)$ and the equality (4.10) remains unchanged.*

Remark 10. *If one is in the case where $a_{1_1,\cdots,i_{2k+1}}\neq 0$, one can rewrite $d_1^\star,d_2^\star,\cdots,d_n^\star$ as $2d_1+1,2d_2,\cdots,2d_n$ and finally rewrite (4.10) as*

$$a_{1_1,\cdots,i_{2k+1}}=\mathbf{E}\left[\frac{1}{V_n}W(\varphi_1)\mathbf{H}_{2d_1+1}\left(W(\varphi_1)\right)\mathbf{H}_{2d_2}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{2d_n}\left(W(\varphi_n)\right)\right].\tag{4.11}$$

Proof : Since $\sum_{r=1}^n d_r^\star=2k+1$, there is an odd number of odd integers in the sequence

$d_r^\star, 1 \leq r \leq n$. Recall that by Lemma 12, we have

$$
\begin{aligned}
a_{i_1,\cdots i_{2k+1}} &= \sum_{u=1}^{n} \mathbf{E}\left(\frac{W(\varphi_u)}{V_n}\prod_{r=1}^{n}\mathbf{H}_{d_r^\star}(W(\varphi_r))\right) \\
&= \mathbf{E}\left[\frac{1}{V_n}W(\varphi_1)\mathbf{H}_{d_1^\star}\left(W(\varphi_1)\right)\mathbf{H}_{d_2^\star}\left(W(\varphi_2)\right)\ldots\mathbf{H}_{d_n^\star}\left(W(\varphi_n)\right)\right] \\
&+ \mathbf{E}\left[\frac{1}{V_n}W(\varphi_2)\mathbf{H}_{d_1^\star}\left(W(\varphi_1)\right)\mathbf{H}_{d_2^\star}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n^\star}\left(W(\varphi_n)\right)\right] \\
&\vdots \\
&+ \mathbf{E}\left[\frac{1}{V_n}W(\varphi_n)\mathbf{H}_{d_1^\star}\left(W(\varphi_1)\right)\mathbf{H}_{d_2^\star}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n^\star}\left(W(\varphi_n)\right)\right]. \quad (4.12)
\end{aligned}
$$

Because of Lemma 14, for each i, the term

$$
\mathbf{E}\left[\frac{1}{V_n}W(\varphi_i)\mathbf{H}_{d_1^\star}\left(W(\varphi_1)\right)\mathbf{H}_{d_2^\star}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n^\star}\left(W(\varphi_n)\right)\right]
$$

is non null if and only if $d_i^\star$ is the only odd integer in $d_r^\star, 1 \leq r \leq n$. Thus, $a_{1_1,\cdots,i_{2k+1}} \neq 0$ if there is only one odd integer in $d_r^\star, 1 \leq r \leq n$. Let $d_i^\star$ with $1 \leq i \leq n$ be this only odd integer. Then, if $j \neq i$, by Lemma 14,

$$
\mathbf{E}\left[\frac{1}{V_n}W(\varphi_j)\mathbf{H}_{d_1^\star}\left(W(\varphi_1)\right)\mathbf{H}_{d_2^\star}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n^\star}\left(W(\varphi_n)\right)\right] = 0.
$$

Thus, using (4.12) yields

$$
a_{1_1,\cdots,i_{2k+1}} = \mathbf{E}\left[\frac{1}{V_n}W(\varphi_i)\mathbf{H}_{d_1^\star}\left(W(\varphi_1)\right)\mathbf{H}_{d_2^\star}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{d_n^\star}\left(W(\varphi_n)\right)\right]
$$

if there is only one odd integer in the sequence $d_r^\star, 1 \leq r \leq n$ and $a_{1_1,\cdots,i_{2k+1}} = 0$ if there is more than one odd integer in the sequence $d_r^\star, 1 \leq r \leq n$. Using the equality in law between $W(\varphi_i)$ and $W(\varphi_1)$, one can perform an index swap ($i \leftrightarrow 1$) to finally obtain the desired result. ■

In the following Lemma, we compute the L^2 norm of F_n. This technical result will be needed in the next section.

Lemma 16. *Let $a_{i_1,\cdots,i_{2k+1}}$ be as given in* (4.9)*. Then, for every $n \in \mathbb{N}$, we have*

$$
\|F_n\|^2_{L^2(\Omega)} = \sum_{k\geq 0}\frac{1}{(2k+1)!}\sum_{i_1,\cdots,i_{2k+1}=1}^{n} a^2_{i_1,\cdots,i_{2k+1}} = 1.
$$

Proof : Firstly, using the isometry of multiple stochastic integrals and the orthogonality of the kernels φ_i, one can write

$$
\begin{aligned}
\mathbf{E}\left(F_n^2\right) &= \sum_{k\geq 0}\left(\frac{1}{(2k+1)!}\right)^2(2k+1)!\sum_{\substack{i_1,\cdots,i_{2k+1}=1\\ j_1,\cdots,j_{2k+1}=1}}^{n} a_{i_1,\cdots,i_{2k+1}}a_{j_1,\cdots,j_{2k+1}} \\
&\times \left\langle \varphi_{i_1}\otimes\cdots\otimes\varphi_{i_{2k+1}}, \varphi_{j_1}\otimes\cdots\otimes\varphi_{j_{2k+1}}\right\rangle_{L^2([0,1]^{4k+2})} \\
&= \sum_{k\geq 0}\frac{1}{(2k+1)!}\sum_{i_1,\cdots,i_{2k+1}=1}^{n} a^2_{i_1,\cdots,i_{2k+1}}.
\end{aligned}
$$

Secondly, using the fact that $F_n^2 = \frac{W_n^2}{V_n^2}$, we have

$$\begin{aligned}\mathbf{E}\left(F_n^2\right) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}e^{-\frac{1}{2}(x_1^2+\cdots+x_n^2)}dx_1\cdots dx_n\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{x_1^2+\cdots+x_n^2}{x_1^2+\cdots+x_n^2}e^{-\frac{1}{2}(x_1^2+\cdots+x_n^2)}dx_1\cdots dx_n = 1\end{aligned}$$

because the mixed terms vanish as in the proof of Lemma 13. ∎

Recall that if X is a Chi-squared random variable with n degrees of freedom (denoted by χ_n^2) then for any $m \geq 0$,

$$\mathbf{E}\left(X^m\right) = 2^m\frac{\Gamma(m+\frac{n}{2})}{\Gamma(\frac{n}{2})}.$$

where $\Gamma(\cdot)$ denotes the standard Gamma function.

When $k = 0$, the coefficients $a_{i_1,\cdots,i_{2k+1}}$ can be easily computed. Indeed, noticing that V_n^2 has a χ_n^2 distribution, we obtain

$$\sum_{i=1}^n a_i = \mathbf{E}\left(\sum_{i=1}^n\frac{1}{V_n}W(\varphi_i)^2\right) = \mathbf{E}\left((V_n^2)^{\frac{1}{2}}\right) = 2^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+\frac{n}{2})}{\Gamma(\frac{n}{2})}.$$

Since $a_1 = a_2 = \cdots = a_n$ we obtain that for every $i = 1,..,n$

$$a_i = \frac{2^{\frac{1}{2}}}{n}\frac{\Gamma(\frac{1}{2}+\frac{n}{2})}{\Gamma(\frac{n}{2})}.$$

The following lemma is the second key result in our goal of calculating the coefficients. It will be used repeatedly in the sequel.

Lemma 17. *Let $\{a_1, a_2, \cdots a_n\}$ be non-negative numbers. Then it holds that*

$$\begin{aligned}&\mathbf{E}\left(\frac{W(\varphi_1)^{2a_1}W(\varphi_2)^{2a_2}\cdots W(\varphi_n)^{2a_n}}{V_n}\right)\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}2^{a_1+\cdots+a_n+\frac{n-1}{2}}\frac{\Gamma\left(a_1+\cdots+a_n+\frac{n-1}{2}\right)}{\Gamma\left(a_1+\cdots+a_n+\frac{n}{2}\right)}\Gamma\left(a_1+\frac{1}{2}\right)\cdots\Gamma\left(a_n+\frac{1}{2}\right).\end{aligned}$$

Proof : By definition, we have

$$\begin{aligned}&\mathbf{E}\left(\frac{W(\varphi_1)^{2a_1}W(\varphi_2)^{2a_2}\cdots W(\varphi_n)^{2a_n}}{V_n}\right)\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{x_1^{2a_1}x_2^{2a_2}\cdots x_n^{2a_n}}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}e^{-\frac{1}{2}(x_1^2+x_2^2+\cdots+x_n^2)}dx_1dx_2\cdots dx_n\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}I.\end{aligned}$$

To compute the above integral I, we introduce n-dimensional polar coordinates. Set

$$\begin{aligned}&x_1 = r\cos\theta_1\\ &x_j = r\cos\theta_j\prod_{r=1}^{j-1}\sin\theta_r, \quad j = 2,\cdots,n-2\\ &x_{n-1} = r\sin\psi\prod_{r=1}^{n-2}\sin\theta_r, \quad x_n = r\cos\psi\prod_{r=1}^{n-2}\sin\theta_r\end{aligned}$$

with $0 \le r < \infty$, $0 \le \theta_i \le \pi$ and $0 \le \psi \le 2\pi$. It can be easily verified that $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$. The Jacobian of the above transformation is given by

$$J = r^{n-1} \prod_{k=1}^{n-2} \sin^k \theta_{n-1-k}\,.$$

Therefore our integral denoted by I becomes

$$\int_0^\infty r^{2(a_1+\cdots+a_n)+n-2} e^{-\frac{r^2}{2}} dr \int_0^{2\pi} (\sin\psi)^{2a_{n-1}+2a_n} (\cos\psi)^{2a_n} d\psi$$
$$\prod_{k=2}^{n-1} \int_0^{\pi} (\sin\theta_{n-k})^{2a_n+2a_{n-1}+\cdots+2a_{n-k+1}+k-1} (\cos\theta_{n-k})^{2a_{n-k}} d\theta_{n-k}.$$

Let us compute the first integral with respect to dr. Using the change of variables $\frac{r^2}{2} = y$, we get

$$\begin{aligned}
\int_0^\infty r^{2(a_1+\cdots+a_n)+n-2} e^{-\frac{r^2}{2}} dr &= 2^{a_1+\cdots+a_n+\frac{n-1}{2}-1} \int_0^\infty dy y^{a_1+\cdots+a_n+\frac{n-1}{2}-1} e^{-y} \\
&= 2^{a_1+\cdots+a_n+\frac{n-1}{2}-1} \Gamma\left(a_1 + \cdots + a_n + \frac{n-1}{2}\right).
\end{aligned}$$

Let us now compute the integral with respect to $d\psi$. We use the following formula : for every $a, b \in \mathbb{Z}$, it holds that

$$\begin{aligned}
\int_0^{2\pi} (\sin\theta)^a (\cos\theta)^b d\theta &= 2\beta\left(\tfrac{a+1}{2}, \tfrac{b+1}{2}\right) \text{ if } m \text{ and } n \text{ are even} \\
&= 0, \qquad \text{if } m \text{ or } n \text{ are odd.}
\end{aligned}$$

This implies that

$$\int_0^{2\pi} (\sin\psi)^{2a_{n-1}+2a_n} (\cos\psi)^{2a_n} d\psi = 2\beta\left(a_n + \frac{1}{2}, a_{n-1} + \frac{1}{2}\right).$$

Finally, we deal with the integral with respect to $d\theta_i$ for $i = 1$ to $n-2$. Using the fact that, for $a, b > -1$, it holds that

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^a (\cos\theta)^b d\theta = \frac{1}{2}\beta\left(\frac{a+1}{2}, \frac{b+1}{2}\right)$$

yields

$$\begin{aligned}
&\int_0^{\pi} (\sin\theta_{n-k})^{2a_n+2a_{n-1}+\cdots+2a_{n-k+1}+k-1} (\cos\theta_{n-k})^{2a_{n-k}} d\theta_{n-k} \\
=\ &\int_0^{\frac{\pi}{2}} (\sin\theta_{n-k})^{2a_n+2a_{n-1}+\cdots+2a_{n-k+1}+k-1} (\cos\theta_{n-k})^{2a_{n-k}} d\theta_{n-k} \\
&+ \int_{\frac{\pi}{2}}^{\pi} (\sin\theta_{n-k})^{2a_n+2a_{n-1}+\cdots+2a_{n-k+1}+k-1} (\cos\theta_{n-k})^{2a_{n-k}} d\theta_{n-k} \\
=\ &\frac{1}{2}\beta\left(a_n + \cdots + a_{n-k+1} + \frac{k}{2}, a_{n-k} + \frac{1}{2}\right) \\
&+ \int_0^{\frac{\pi}{2}} (\sin(\theta_{n-k} + \frac{\pi}{2}))^{2a_n+2a_{n-1}+\cdots+2a_{n-k+1}+k-1} (\cos(\theta_{n-k} + \frac{\pi}{2}))^{2a_{n-k}} d\theta_{n-k} \\
=\ &\beta\left(a_n + \cdots + a_{n-k+1} + \frac{k}{2}, a_{n-k} + \frac{1}{2}\right)
\end{aligned}$$

because $\sin(\theta+\frac{\pi}{2}) = \cos\theta$ and $\cos(\theta+\frac{\pi}{2}) = -\sin(\theta)$. By gathering the above calculations, the integral I becomes

$$\begin{aligned}
I &= 2^{a_1+\cdots+a_n+\frac{n-1}{2}}\Gamma\left(a_1+\cdots+a_n+\frac{n-1}{2}\right)\beta\left(a_n+\frac{1}{2},a_{n-1}+\frac{1}{2}\right)\\
&\quad\times\prod_{k=2}^{n-1}\beta\left(a_n+\cdots+a_{n-k+1}+\frac{k}{2},a_{n-k}+\frac{1}{2}\right)\\
&= 2^{a_1+\cdots+a_n+\frac{n-1}{2}}\Gamma\left(a_1+\cdots+a_n+\frac{n-1}{2}\right)\frac{\Gamma\left(a_n+\frac{1}{2}\right)\Gamma\left(a_{n-1}+\frac{1}{2}\right)}{\Gamma\left(a_n+a_{n-1}+1\right)}\\
&\quad\times\prod_{k=2}^{n-1}\frac{\Gamma\left(a_n+\cdots+a_{n-k+1}+\frac{k}{2}\right)\Gamma\left(a_{n-k}+\frac{1}{2}\right)}{\Gamma\left(a_n+a_{n-1}+\cdots+a_{n-k}+\frac{k+1}{2}\right)}\\
&= 2^{a_1+\cdots+a_n+\frac{n-1}{2}}\frac{\Gamma\left(a_1+\cdots+a_n+\frac{n-1}{2}\right)}{\Gamma\left(a_1+\cdots+a_n+\frac{n}{2}\right)}\Gamma\left(a_1+\frac{1}{2}\right)\cdots\Gamma\left(a_n+\frac{1}{2}\right).
\end{aligned}$$

This concludes the proof. ■

The following result gives the asymptotic behavior of the coefficients when $n\to\infty$.

Lemma 18. *For every $1\le i_1,\cdots,i_{2k+1}\le n$, let $a_{i_1,\cdots,i_{2k+1}}$ be as defined in (4.7). As in (4.11), let $2d_1+1,2d_2,\cdots,2d_r,\cdots,2d_n$ denote the number of times the integer r appears in the sequence $\{i_1,i_2,\cdots,i_{2k+1}\}$ with $\sum_{r=1}^n d_r = k$. Then when $n\to\infty$,*

$$\begin{aligned}
a_{i_1,\cdots,i_{2k+1}} &\sim \frac{1}{k!}(2k-1)!!\frac{(2d_1+1)!(2d_2)!\cdots(2d_n)!}{(d_1!d_2!\cdots d_n!)^2}\\
&\quad\times 2^{-2k}(-1)^k\left(\prod_{j=0}^{n}\sum_{l_j=0}^{d_j}(-1)^{l_j}C_{d_j}^{l_j}l_j^{d_j}\right)\frac{1}{n^{\frac{1}{2}+|A|}}
\end{aligned}\qquad(4.13)$$

where

$$A := \{2d_1+1,2d_2,\cdots,2d_n\}\setminus\{0,1\}$$

and $|A|$ is the cardinal of A.

Proof : We recall the following explicit formula for the Hermite polynomials

$$\mathbf{H}_d(x) = d!\sum_{l=0}^{[\frac{d}{2}]}\frac{(-1)^l}{2^l l!(d-2l)!}x^{d-2l}.\qquad(4.14)$$

Using (4.14) and (4.11) we can write

$$\begin{aligned}
a_{i_1,\cdots,i_{2k+1}} &= \mathbf{E}\left[\frac{1}{V_n}W(\varphi_1)\mathbf{H}_{2d_1+1}\left(W(\varphi_1)\right)\mathbf{H}_{2d_2}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{2d_n}\left(W(\varphi_n)\right)\right]\\
&= (2d_1+1)!(2d_2)!\cdots(2d_n)!\sum_{l_1=0}^{d_1}\sum_{l_2=0}^{d_2}\cdots\sum_{l_n=0}^{d_n}\frac{(-1)^{l_1+l_2+\cdots+l_n}}{2^{l_1+l_2+\cdots+l_n}l_1!\cdots l_n!}\\
&\quad\times\frac{\mathbf{E}\left[\frac{1}{V_n}W(\varphi_1)^{2d_1+2-2l_1}W(\varphi_2)^{2d_2-2l_2}\cdots W(\varphi_n)^{2d_n-2l_n}\right]}{(2d_1+1-2l_2)!(2d_2-2l_2)!\cdots(2d_n-2l_n)!}.
\end{aligned}$$

At this point, we use Lemma 17 to rewrite the expectation in the last equation.

$$
\begin{aligned}
& \mathbf{E}\left[\frac{1}{V_n} W(\varphi_1)\mathbf{H}_{2d_1+1}\left(W(\varphi_1)\right)\mathbf{H}_{2d_2}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{2d_n}\left(W(\varphi_n)\right)\right] \\
= \; & (2d_1+1)!(2d_2)!\cdots(2d_n)! \sum_{l_1=0}^{d_1}\sum_{l_2=0}^{d_2}\cdots\sum_{l_n=0}^{d_n} \frac{(-1)^{l_1+l_2+\cdots+l_n}}{2^{l_1+l_2+\cdots+l_n} l_1!\cdots l_n!} \\
& \times \frac{2^{d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n-1}{2}}}{(2\pi)^{\frac{n}{2}}(2d_1+1-2l_2)!(2d_2-2l_2)!\cdots(2d_n-2l_n)!} \\
& \times \frac{\Gamma\left(d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n-1}{2}\right)}{\Gamma\left(d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n}{2}\right)} \\
& \times \Gamma\left(d_1+1-l_1+\frac{1}{2}\right)\Gamma\left(d_2-l_2+\frac{1}{2}\right)\cdots\Gamma\left(d_n-l_n+\frac{1}{2}\right) \\
= \; & (2d_1+1)!(2d_2)!\cdots(2d_n)! \sum_{l_1=0}^{d_1}\sum_{l_2=0}^{d_2}\cdots\sum_{l_n=0}^{d_n} \frac{(-1)^{l_1+l_2+\cdots+l_n}}{2^{2(l_1+l_2+\cdots+l_n)} l_1!\cdots l_n!} \\
& \times \frac{2^{d_1+1+d_2+\cdots+d_n-\frac{1}{2}}}{\pi^{\frac{n}{2}}(2d_1+1-2l_2)!(2d_2-2l_2)!\cdots(2d_n-2l_n)!} \\
& \times \frac{\Gamma\left(d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n-1}{2}\right)}{\Gamma\left(d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n}{2}\right)} \\
& \times \Gamma\left(d_1+1-l_1+\frac{1}{2}\right)\Gamma\left(d_2-l_2+\frac{1}{2}\right)\cdots\Gamma\left(d_n-l_n+\frac{1}{2}\right).
\end{aligned}
$$

We claim that for any integers $d \geq l$,

$$
\frac{(-1)^l}{2^{-2l} l!(2d-2l)!}\Gamma\left(d-l+\frac{1}{2}\right) = \sqrt{\pi}\frac{2^{-2d}(-1)^l}{d!}C_d^l. \tag{4.15}
$$

Recall the relation satisfied by the Gamma function : for every $z > 0$,

$$
\Gamma(z+1) = z\Gamma(z) \text{ and } \Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}2^{1-2z}\Gamma(2z). \tag{4.16}
$$

Then

$$
\begin{aligned}
\frac{(-1)^l}{2^{-2l} l!(2d-2l)!}\Gamma\left(d-l+\frac{1}{2}\right) &= \frac{(-1)^l}{2^{-2l} l!(2d-2l)!}\frac{\Gamma\left(d-l+1+\frac{1}{2}\right)}{d-l-\frac{1}{2}} \\
&= \frac{(-1)^l}{2^{-2l} l!(2d-2l)!}\frac{\Gamma(2d-2l+2)}{\Gamma(d-l+1)}\sqrt{\pi}2^{1-2(d-l+1)} \\
&= \sqrt{\pi}2^{-2d}\frac{(-1)^l}{l!(2d-2l)!}\frac{(2d-2l+1)!}{(d-l)!(2d-2l+1)} \\
&= \sqrt{\pi}\frac{2^{-2d}(-1)^l}{d!}C_d^l
\end{aligned}
$$

and (4.15) is proved. In the same way, using only the second relation in (4.16), we obtain

$$
\frac{(-1)^{l_1}}{2^{-2l_1} l_1!(2d_1+1-2l_1)!}\Gamma\left(d_1+1-l_1+\frac{1}{2}\right) = \sqrt{\pi}\frac{2^{-1-2d_1}(-1)^{l_1}}{d_1!}C_{d_1}^{l_1}. \tag{4.17}
$$

Putting together (4.15) and (4.17) we find

$$\begin{aligned}
&\mathbf{E}\left[\frac{1}{V_n}W(\varphi_1)\mathbf{H}_{2d_1+1}\left(W(\varphi_1)\right)\mathbf{H}_{2d_2}\left(W(\varphi_2)\right)\cdots\mathbf{H}_{2d_n}\left(W(\varphi_n)\right)\right]\\
=\ &\frac{(2d_1+1)!(2d_2)!\cdots(2d_n)!}{d_1!d_2!\cdots d_n!}2^{-(d_1+\cdots+d_n)-\frac{1}{2}}\sum_{l_1=0}^{d_1}\sum_{l_2=0}^{d_2}\cdots\sum_{l_n=0}^{d_n}(-1)^{l_1+l_2+\cdots+l_n}C_{d_1}^{l_1}\cdots C_{d_n}^{l_n}\\
&\times\frac{\Gamma\left(d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n-1}{2}\right)}{\Gamma\left(d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n}{2}\right)}.
\end{aligned}$$

By Stirling's formula, when n goes to infinity, we have

$$\frac{\Gamma\left(d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n-1}{2}\right)}{\Gamma\left(d_1+1+d_2+\cdots+d_n-(l_1+l_2+\cdots+l_n)+\frac{n}{2}\right)}\sim\frac{1}{\sqrt{k+1-(l_1+\cdots+l_n)+\frac{n}{2}}}.$$

Therefore we need to study the behavior of the sequence

$$t_n:=\sum_{l_1=0}^{d_1}\sum_{l_2=0}^{d_2}\cdots\sum_{l_n=0}^{d_n}(-1)^{l_1+l_2+\cdots+l_n}C_{d_1}^{l_1}\cdots C_{d_n}^{l_n}\frac{1}{\sqrt{k+1-(l_1+\cdots+l_n)+\frac{n}{2}}}$$

as $n\to\infty$. We can write

$$t_n=\frac{1}{\sqrt{n}}\sqrt{2}g(\frac{1}{n})$$

where

$$g(x)=\sum_{l_1=0}^{d_1}\sum_{l_2=0}^{d_2}\cdots\sum_{l_n=0}^{d_n}(-1)^{l_1+l_2+\cdots+l_n}C_{d_1}^{l_1}\cdots C_{d_n}^{l_n}\frac{1}{\sqrt{2k+2-(l_1+\cdots+l_n)x+1}}.$$

Since for every $d\geq 1$

$$\sum_{l=0}^{d}(-1)^lC_d^l=0$$

we clearly have $g(0)=0$. The q^{th} derivative of g at zero is

$$g^{(q)}(0)=(-1)^q\frac{(2q-1)!!}{2^q}\left[2k+2-(l_1+\cdots+l_n)\right]^q.$$

Repeatedly using the relation $C_n^k=\frac{n}{k}C_{n-1}^{k-1}$ we can prove that

$$\sum_{l=0}^{d}(-1)^lC_d^l l^q=0$$

for every $q=0,1,\cdots,d-1$. Therefore the first non-zero term in the Taylor decomposition of the function g around zero is

$$\sum_{l_1=0}^{d_1}\sum_{l_2=0}^{d_2}\cdots\sum_{l_n=0}^{d_n}(-1)^{l_1+l_2+\cdots+l_n}C_{d_1}^{l_1}\cdots C_{d_n}^{l_n}l_1^{d_1}\cdots l_n^{d_n}$$

which appears when we take the derivative of order $d_1+d_2+\cdots+d_n$. We obtain that, for x close to zero,

$$g(x)\sim(-1)^{d_1+\cdots+d_n}\frac{(2(d_1+\cdots+d_n)-1)!!}{2^{d_1+\cdots+d_n}}\prod_{j=0}^{n}\sum_{l_j=0}^{d_j}(-1)^{l_j}C_{d_j}^{l_j}l_j^{d_j}\times H(d_1,\cdots,d_n)x^{|A|}$$

where

$$A = \{d_1, \cdots, d_n\} \setminus \{0\} = \{2d_1+1, 2d_2, \cdots, 2d_n\} \setminus \{0,1\}$$

and $H(d_1, \cdots, d_n)$ is the coefficient of $l_1^{d_1} \cdots l_n^{d_n}$ in the expansion of $(l_1 + \cdots + l_n)^{d_1+\cdots+d_n}$. That is

$$H(d_1, \cdots, d_n) = C^{d_1}_{d_1+\cdots+d_n} C^{d_2}_{d_2+\cdots+d_n} \cdots C^{d_{n-1}}_{d_{n-1}+d_n} = \frac{(d_1+\cdots+d_n)!}{d_1! \cdots d_n!}.$$

We finally have

$$\begin{aligned}
a_{i_1,\cdots,i_{2k+1}} &= \frac{(2d_1+1)!(2d_2)!\cdots(2d_n)!}{(d_1!d_2!\cdots d_n!)^2} 2^{-(d_1+\cdots+d_n)} (-1)^{d_1+\cdots+d_n} \frac{(2(d_1+\cdots+d_n)-1)!!}{2^{d_1+\cdots+d_n}} \\
&\quad \times \left(\prod_{j=0}^{n} \sum_{l_j=0}^{d_j} (-1)^{l_j} C^{l_j}_{d_j} l_j^{d_j} \right) \frac{(d_1+\cdots+d_n)!}{d_1!\cdots d_n!} \frac{1}{n^{\frac{1}{2}+|A|}} \\
&= k!(2k-1)!! \frac{(2d_1+1)!(2d_2)!\cdots(2d_n)!}{(d_1!d_2!\cdots d_n!)^2} 2^{-2k}(-1)^k \left(\prod_{j=0}^{n} \sum_{l_j=0}^{d_j} (-1)^{l_j} C^{l_j}_{d_j} l_j^{d_j} \right) \frac{1}{n^{\frac{1}{2}+|A|}} \\
. &= k!(2k-1)!! \frac{(2d_1+1)!(2d_2)!\cdots(2d_n)!}{(d_1!d_2!\cdots d_n!)^2} 2^{-2k}(-1)^k \left(\prod_{j=0}^{n} t(d_j) \right) \frac{1}{n^{\frac{1}{2}+|A|}}
\end{aligned}$$

with for $i = 1, \cdots, n$

$$t(d_j) := \sum_{l_j=0}^{d_j} (-1)^{l_j} C^{l_j}_{d_j} l_j^{d_j}. \tag{4.18}$$

■

4.4 Computation of the Berry-Esséen bound

Let us first recall the following result (see [dlPLS09], page 53) : for $2 < p \leq 3$,

$$\sup_{z \in \mathbb{R}} |P(F_n \leq z) - \Phi(z)| \leq 25\mathbf{E}\left(|Z|^p\right) n^{1-\frac{p}{2}} \tag{4.19}$$

where Z is a standard normal random variable and Φ is its repartition function. In particular for $p = 3$ we get

$$\sup_{z \in \mathbb{R}} |P(F_n \leq z) - \Phi(z)| \leq 25\mathbf{E}\left(|Z|^3\right) n^{-\frac{1}{2}}.$$

We now compute the Berry-Essen bound obtained via Malliavin calculus in order to compare it with (4.19). Formula (4.9) yields

$$D_\alpha F_n = \sum_{k \geq 0} \frac{2k+1}{(2k+1)!} \sum_{i_1,\cdots,i_{2k+1}=1}^{n} a_{i_1,\cdots,i_{2k+1}} I_{2k}\left((\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_{2k=1}})^{\sim} \right)(\cdot, \alpha) \tag{4.20}$$

(here $(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_{2k=1}})^{\sim}$ denotes the symmetrization of the function $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ with respect to its k variables) and

$$D_\alpha (-L)^{-1} F_n = \sum_{k} \frac{1}{(2k+1)!} \sum_{i_1,\cdots,i_{2k+1}=1}^{n} a_{i_1,\cdots,i_{2k+1}} I_{2k}\left((\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_{2k+1}})^{\sim} \right)(\cdot, \alpha). \tag{4.21}$$

It is now possible to calculate the quantity

$$\mathbf{E}\left(1-\langle DF_n, D(-L)^{-1}F_n\rangle\right)^2$$

more explicitly by using the product formula (4.3) and the isometry of multiple stochastic integrals.

Lemma 19. *For every $n \geq 2$,*

$$\begin{aligned}
&\mathbf{E}\left(1-\langle DF_n, D(-L)^{-1}F_n\rangle\right)^2\\
=\ &\sum_{m\geq 1}(2m)!\sum_{i_1,\cdots,i_{2m}=1}^{n}\left(\sum_{k=0}^{2m}\frac{1}{k!}\frac{1}{(2m-k)!}\sum_{r\geq 0}\frac{1}{r!}\frac{1}{2m-k+r+1}\sum_{u_1,\cdots,u_{r+1}=1}^{n}\right.\\
&\left. a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k}a_{u_1,u_2,\cdots,u_{r+1},i_{k+1},\cdots,i_{2m}}\right)^2.
\end{aligned}$$

Proof : Using (4.20) and (4.21), we can calculate the following quantity.

$$\begin{aligned}
\langle DF_n, D(-L)^{-1}F_n\rangle &= \sum_{k,l\geq 0}\frac{1}{(2k)!}\frac{1}{(2l+1)!}\sum_{i_1,\cdots,i_{2k+1}=1}^{n}a_{i_1,\cdots,i_{2k+1}}\sum_{j_1,\cdots,j_{2l+1}=1}^{n}a_{j_1,\cdots,j_{2l+1}}\\
&\quad\times\int_0^\infty d\alpha I_{2k}\left((\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_{2k+1}})^{\sim}\right)(\cdot,\alpha)I_{2l}\left((\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_{2l+1}})^{\sim}\right)(\cdot,\alpha)\\
&= \sum_{k,l\geq 0}\frac{1}{(2k)!}\frac{1}{(2l+1)!}\sum_{u=1}^{n}\sum_{i_1,\cdots,i_{2k}=1}^{n}a_{u,i_1,\cdots,i_{2k}}\sum_{j_1,\cdots,j_{2l}=1}^{n}a_{u,j_1,\cdots,j_{2l}}\\
&\quad\times I_{2k}\left((\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_{2k}})\right)I_{2l}\left((\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_{2j}})\right).
\end{aligned}$$

The product formula (4.3) applied to the last equality yields

$$\begin{aligned}
&\sum_{i_1,\cdots,i_{2k}=1}^{n}a_{u,i_1,\cdots,i_{2k}}\sum_{j_1,\cdots,j_{2l}=1}^{n}a_{u,j_1,\cdots,j_{2l}}I_{2k}\left((\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_{2k}})\right)I_{2l}\left((\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_{2j}})\right)\\
=\ &\sum_{r=0}^{(2k)\wedge(2l)}r!C_{2k}^rC_{2l}^r\sum_{u_1,\cdots,u_r=1}^{n}\sum_{i_1,\cdots,i_{2k-r}=1}^{n}\sum_{j_1,\cdots,j_{2l-r}=1}^{n}a_{u,u_1,\cdots,u_r,i_1,\cdots,i_{2k-r}}a_{u,u_1,\cdots,u_r,j_1,\cdots,j_{2l-r}}\\
&\times I_{2k+2l-2r}\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_{2k-r}}\otimes\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_{2l-r}}\right)
\end{aligned}$$

and therefore we obtain

$$\begin{aligned}
&\langle DF_n, D(-L)^{-1}F_n\rangle\\
=\ &\sum_{k,l\geq 0}\frac{1}{(2k)!}\frac{1}{(2l+1)!}\sum_{r=0}^{(2k)\wedge(2l)}r!C_{2k}^rC_{2l}^r\\
&\times\sum_{u_1,\cdots,u_{r+1}=1}^{n}\sum_{i_1,\cdots,i_{2k-r}=1}^{n}\sum_{j_1,\cdots,j_{2l-r}=1}^{n}a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_{2k-r}}a_{u_1,u_2,\cdots,u_{r+1},j_1,\cdots,j_{2l-r}}\\
&\times I_{2k+2l-2r}\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_{2k-r}}\otimes\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_{2l-r}}\right).
\end{aligned}\tag{4.22}$$

Remark 11. *The chaos of order zero in the above expression is obtained for $k=l$ and $r=2k$. It is therefore equal to*

$$\sum_{k\geq 0}\frac{1}{(2k)!}\frac{1}{(2k+1)!}(2k)!\sum_{i_1,\cdots,i_{2k+1}=1}^{n}a_{i_1,\cdots,i_{2k+1}}^2$$

which is also equal to 1 as follows from Lemma 16. Therefore it will vanish when we consider the difference $1-\langle DF_n, D(-L)^{-1}F_n\rangle$. This difference will have only chaoses of even orders.

By changing the order of summation and by using the changes of indices $2k-r=k'$ and $2l-r=l'$, we can write

$$\begin{aligned}
&\langle DF_n, D(-L)^{-1}F_n\rangle \\
&= \sum_{r\geq 0} r! \sum_{2k\geq r}\sum_{2l\geq r} \frac{1}{(2k)!}\frac{1}{(2l+1)!} C_{2k}^r C_{2l}^r \\
&\times \sum_{u_1,\cdots,u_{r+1}=1}^{n} \sum_{i_1,\cdots,i_{2k-r}=1}^{n} \sum_{j_1,\cdots,j_{2l-r}=1}^{n} a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_{2k-r}} a_{u_1,u_2,\cdots,u_{r+1},j_1,\cdots,j_{2l-r}} \\
&\times I_{2k+2l-2r}\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_{2k-r}}\otimes\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_{2l-r}}\right) \\
= &\sum_{r\geq 0}\sum_{k,l\geq 0} \frac{1}{(k+r)!}\frac{1}{(l+r+1)!} C_{k+r}^r C_{l+r}^r \\
&\times \sum_{u_1,\cdots,u_{r+1}=1}^{n} \sum_{i_1,\cdots,i_k=1}^{n} \sum_{j_1,\cdots,j_l=1}^{n} a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k} a_{u_1,u_2,\cdots,u_{r+1},j_1,\cdots,j_l} \\
&\times I_{2k+2l-2r}\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\otimes\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_l}\right) \\
= &\sum_{k,l\geq 0}\sum_{r\geq 0} r!\frac{1}{(k+r)!}\frac{1}{(l+r+1)!} C_{k+r}^r C_{l+r}^r \\
&\times \sum_{u_1,\cdots,u_{r+1}=1}^{n} \sum_{i_1,\cdots,i_k=1}^{n} \sum_{j_1,\cdots,j_l=1}^{n} a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k} a_{u_1,u_2,\cdots,u_{r+1},j_1,\cdots,j_l} \\
&\times I_{k+l}\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\otimes\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_l}\right).
\end{aligned}$$

Once again using a change of indices ($k+l=m$), we obtain

$$\begin{aligned}
&\langle DF_n, D(-L)^{-1}F_n\rangle \\
= &\sum_{m\geq 0}\sum_{k=0}^{m}\sum_{r\geq 0} r!\frac{1}{(k+r)!}\frac{1}{(m-k+r+1)!} C_{k+r}^r C_{m-k+r}^r \\
&\times \sum_{u_1,\cdots,u_{r+1}=1}^{n} \sum_{i_1,\cdots,i_k=1}^{n} \sum_{j_1,\cdots,j_{m-k}=1}^{n} a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k} a_{u_1,u_2,\cdots,u_{r+1},j_1,\cdots,j_{m-k}} \\
&\times I_m\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\otimes\varphi_{j_1}\otimes\cdots\otimes\varphi_{j_{m-k}}\right) \\
= &\sum_{m\geq 0}\sum_{k=0}^{m}\frac{1}{k!}\frac{1}{(m-k)!}\sum_{r\geq 0}\frac{1}{r!}\frac{1}{m-k+r+1} \sum_{u_1,\cdots,u_{r+1}=1}^{n} \sum_{i_1,\cdots,i_m=1}^{n} \\
&\times a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k} a_{u_1,u_2,\cdots,u_{r+1},i_{k+1},\cdots,i_m} I_m\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\otimes\varphi_{i_{k+1}}\otimes\cdots\otimes\varphi_{i_m}\right)
\end{aligned}$$

where at the end we renamed the indices $j_1,\cdots,j_{m-m}$ as $i_{k+1},\cdots,i_m$. We obtain

$$\langle DF_n, D(-L)^{-1}F_n\rangle = \sum_{m\geq 0} I_m(h_m^{(n)})$$

where

$$\begin{aligned}
h_m^{(n)} \;=\; &\sum_{k=0}^{m}\frac{1}{k!}\frac{1}{(m-k)!}\sum_{r\geq 0}\frac{1}{r!}\frac{1}{m-k+r+1} \sum_{u_1,\cdots,u_{r+1}=1}^{n} \sum_{i_1,\cdots,i_m=1}^{n} \\
&a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k} a_{u_1,u_2,\cdots,u_{r+1},i_{k+1},\cdots,i_m} \\
&\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\otimes\varphi_{i_{k+1}}\otimes\cdots\otimes\varphi_{i_m} \qquad (4.23)
\end{aligned}$$

Let us make some comments about this result before going any further. These remarks will simplify the expression that we have just obtained. As follows from Lemma 12, the coefficients $a_{i_1,\cdots,i_k}$ are zero if k is even. Therefore, the numbers $r+1+k$ and $r+1+m-k$ must be odd. This implies that m must be even and this is coherent with our previous observation (see Remark 11) that the chaos expansion of $\langle DF_n, D(-L)^{-1}F_n\rangle$ only contains chaoses of even orders. The second comment concerns the chaos of order zero. If $m=0$ then $k=0$ and we obtain

$$h_0^{(n)} = \sum_{r\geq 0}\sum_{u_1,\cdots,u_{r+1}=1}^{n} \frac{1}{r!}\frac{1}{r+1} a^2_{u_1,\cdots,u_{r+1}} = \sum_{r\geq 1}\frac{1}{r!}\sum_{u_1,\cdots,u_r=1}^{n} a^2_{u_1,\cdots,u_r}.$$

Thus, because the summand $\sum_{r\geq 1}\frac{1}{r!}\sum_{u_1,\cdots,u_r=1}^{n} a^2_{u_1,\cdots,u_r} - 1$ is zero by using Lemma 16,

$$\begin{aligned}\langle DF_n, D(-L)^{-1}F_n\rangle - 1 &= \left(\sum_{r\geq 1}\frac{1}{r!}\sum_{u_1,\cdots,u_r=1}^{n} a^2_{u_1,\cdots,u_r} - 1\right) + \sum_{m\geq 1} I_{2m}(h_{2m}^{(n)}) \\ &= \sum_{m\geq 1} I_{2m}(h_{2m}^{(n)})\end{aligned}$$

with $h_{2m}^{(n)}$ given by (4.23).

Using the isometry formula of multiple integrals in order to compute the L^2 norm of the above expression and noticing that the function $h_{2m}^{(n)}$ is symmetric, we find that

$$\mathbf{E}\left(\left(\langle DF_n, D(-L)^{-1}F_n\rangle - 1\right)^2\right) = \sum_{m\geq 1}(2m)!\langle h_{2m}^{(n)}, h_{2m}^{(n)}\rangle_{L^2([0,1]^{2m})}$$

$$\begin{aligned}&= \sum_{m\geq 1}(2m)!\sum_{k,l=0}^{2m}\frac{1}{k!}\frac{1}{l!}\frac{1}{(2m-k)!}\frac{1}{(2m-l)!}\sum_{r,q\geq 0}\frac{1}{r!}\frac{1}{q!}\frac{1}{2m-k+r+1}\frac{1}{2m-l+q+1} \\ &\quad\times \sum_{u_1,\cdots,u_{r+1}=1}^{n}\sum_{v_1,\cdots,v_{q+1}=1}^{n}\sum_{i_1,\cdots,i_{2m}=1}^{n} \\ &\quad\times a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k} a_{u_1,u_2,\cdots,u_{r+1},i_{k+1},\cdots,i_{2m}} a_{v_1,v_2,\cdots,v_{q+1},i_1,\cdots,i_k} a_{v_1,v_2,\cdots,v_{q+1},i_{k+1},\cdots,i_{2m}} \\ &= \sum_{m\geq 1}(2m)!\sum_{i_1,\cdots,i_{2m}=1}^{n}\left(\sum_{k=0}^{2m}\frac{1}{k!}\frac{1}{(2m-k)!}\sum_{r\geq 0}\frac{1}{r!}\frac{1}{2m-k+r+1}\sum_{u_1,\cdots,u_{r+1}=1}^{n}\right. \\ &\quad\left. a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k} a_{u_1,u_2,\cdots,u_{r+1},i_{k+1},\cdots,i_{2m}}\right)^2,\end{aligned}$$

which is the desired result. ■

Before proving our main result, let us discuss a particular case as an exemple in order to better understand the general phenomenon. This is both useful and important in order to have a good overview of the functioning of a simple case. Assume that $k=0$ and $l=1$. The corresponding summand in (4.22) reduces to

$$\frac{1}{3!}\sum_{u=1}^{n} a_u \sum_{j_1,j_2=1}^{n} a_{u,j_1,j_2} I_2\left(\varphi_{j_1}\otimes\varphi_{j_2}\right).$$

Its L^2-norm is

$$\frac{1}{3}\sum_{j_1,j_2=1}^{n}\left(\sum_{u=1}^{n} a_u a_{u,j_1,j_2}\right)^2 = \frac{1}{3}\sum_{j_1=1}^{n}\left(\sum_{u=1}^{n} a_u a_{u,j_1,j_1}\right)^2$$

because $a_{u,j_1,j_2} = 0$ if $j_1 \neq j_2$. Using (4.13), it reduces to a quantity equivalent to

$$\frac{1}{3}(na_1^2 a_{1,1,1}^2 + n((n-1)a_1 a_{1,1,2})^2)$$

which, using (4.13) again, is of order

$$n\left(\frac{1}{\sqrt{n}}\right)^2\left(\frac{1}{n^{\frac{3}{2}}}\right)^2 + n\left((n-1)\frac{1}{\sqrt{n}}\frac{1}{n^{\frac{3}{2}}}\right)^2 \sim n^{-1}.$$

The following theorem, which gathers all of the previous results of the paper, is the general equivalent of the toy exemple presented above.

Theorem 13. *For any integer $n \geq 2$,*

$$\mathbf{E}\left(\left(\langle DF_n, D(-L)^{-1}F_n\rangle - 1\right)^2\right) \leq \frac{c_0}{n}$$

with

$$\begin{aligned} c_0 &= \sum_{m\geq 1}(2m)!\left(\sum_{k=0}^{2m}\frac{1}{2k!}\frac{1}{(2m-2k)!}\sum_{r\geq 0}\frac{1}{(2r)!}\frac{1}{2m-2k+2r+1}c(k,r,m)\right)^2 \qquad (4.24)\\ &+\left(\sum_{k=0}^{2m}\frac{1}{(2k+1)!}\frac{1}{(2m-2k-1)!}\sum_{r\geq 0}\frac{1}{(2r-1)!}\frac{1}{2m-2k+2r+1}c(k,r,m)\right)^2\end{aligned}$$

and where $c(k,r,m)$ is given by (4.27).

Proof : Observe that the integers $r+1+k$ and $r+1+2m-k$ both have to be odd numbers (otherwise the coefficients $a_{u_1,u_2,\cdots,u_{r+1},i_1,\cdots,i_k}$ and $a_{u_1,u_2,\cdots,u_{r+1},i_{k+1},\cdots,i_{2m}}$ vanish). This implies two cases : either r is even and k is even or r is odd and k is odd. Thus, we can write

$$\begin{aligned} &\mathbf{E}\left(\left(\langle DF_n, D(-L)^{-1}F_n\rangle - 1\right)^2\right)\\ &= \sum_{m\geq 1}(2m)!\sum_{i_1,\cdots,i_{2m}=1}^{n}\left(\sum_{k=0}^{2m}\frac{1}{2k!}\frac{1}{(2m-2k)!}\sum_{r\geq 0}\frac{1}{(2r)!}\frac{1}{2m-2k+2r+1}\sum_{u_1,\cdots,u_{2r+1}=1}^{n}\right.\\ &\qquad \left. a_{u_1,u_2,\cdots,u_{2r+1},i_1,\cdots,i_{2k}}a_{u_1,u_2,\cdots,u_{2r+1},i_{2k+1},\cdots,i_{2m}}\right)^2\\ &+\sum_{m\geq 1}(2m)!\sum_{i_1,\cdots,i_{2m}=1}^{n}\left(\sum_{k=0}^{2m}\frac{1}{(2k+1)!}\frac{1}{(2m-2k-1)!}\sum_{r\geq 0}\frac{1}{(2r-1)!}\frac{1}{2m-2k+2r+1}\right.\\ &\qquad \left.\sum_{u_1,\cdots,u_{2r}=1}^{n}a_{u_1,u_2,\cdots,u_{2r},i_1,\cdots,i_{2k+1}}a_{u_1,u_2,\cdots,u_{2r},i_{2k+2},\cdots,i_{2m}}\right)^2. \qquad (4.25)\end{aligned}$$

Let us treat the first part of the sum (4.25). Assume that the number of common numbers occurring in the sets $\{u_1,\cdots,u_{2r+1}\}$ and $\{i_1,\cdots,i_{2k}\}$ is x and and the number of common numbers occurring in the sets $\{u_1,\cdots,u_{2r+1}\}$ and $\{i_{2k+1},\cdots,i_{2m-2k}\}$ is y. This can be formally written as

$$|\{u_1,\cdots,u_{2r+1}\}\cap\{\{i_1,\cdots,i_{2k}\}| = x$$

and

$$|\{u_1,\cdots,u_{2r+1}\}\cap\{i_{2k+1},\cdots,i_{2m-2k}\}| = y.$$

It is clear that

$$x \leq (2r+1) \wedge 2k \text{ and } y \leq (2r+1) \wedge 2m - 2k.$$

This also implies $x + y \leq 2m$. According to the definitions of x and y, it can be observed that x and y must be even. We will denote them by $2x$ and $2y$ from now on.
The next step in the proof is to determine how many distinct sequences of numbers can occur in the set

$$\{u_1, \cdots, u_{2r+1}, i_1, \cdots, i_{2k}\}.$$

We can have sequences of lengths (all of the lengths that we consider from now on are greater or equal to one) $2c_1, 2c_2, \cdots, 2c_{l_1}$ with $2(c_1 + \cdots + c_{l_1}) = 2x$ in the set $\{u_1, \cdots, u_{2r+1}\} \cap \{i_1, \cdots, i_{2k}\}$ but also sequences of lengths $2d_1, 2d_2, \cdots, 2d_{l_2}$ with $2(d_1 + \cdots + d_{l_2}) = 2k - 2x$ in the set $\{i_1, \cdots, i_{2k}\} \setminus \{u_1, \cdots, u_{2r+1}\}$ as well as sequences of lengths $2e_1 + 1, 2e_2, \cdots, 2e_{l_3}$ with $1 + 2(e_1 + \cdots + e_{l_3}) = 2r + 1 - 2x$ in the set $\{u_1, \cdots, u_{2r+1}\} \setminus \{i_1, \cdots, i_{2k}\}$. In this last sequence we have one (and only one) length equal to 1 (because we are allowed to choose only one odd number in the set $\{u_1, \cdots, u_{2r+1}\} \setminus \{i_1, \cdots, i_{2k}\}$). We will have, if we have a configuration as above,

$$a_{u_1, u_2, \cdots, u_{2r+1}, i_1, \cdots, i_{2k}} \leq c(r, c, e) n^{-\frac{1}{2} - l_1 - l_2 - l_3}$$

where

$$c(r, c, e) = r!(2r-1)!! \frac{(2c_1)! \cdots (2c_{l_1})!(2e_1+1)!(2e_2)! \cdots (2e_{l_3})!}{(c_1! \cdots c_{l_1}! e_1! \cdots e_{l_3}!)^2} t(c_1) \cdots t(c_{l_1}) t(e_1) \cdots t(e_{l_3}) \tag{4.26}$$

and the constants t are given by (4.18).
In the same way, assuming that we have sequences of lengths $2f_1, 2f_2, \cdots, 2f_{l_4}$ with $2(f_1 + \cdots + f_{l_4}) = 2m - 2k - 2y$ in the set $\{i_{2k+1}, \cdots, i_{2m}\} \setminus \{u_1, \cdots, u_{2r+1}\}$ and sequences of lengths $2g_1 + 1, 2g_2, \cdots, 2g_{l_5}$ with $1 + 2(g_1 + \cdots + g_5) = 2r + 1 - 2y$ in the set $\{u_1, \cdots, u_{2r+1}\} \setminus \{i_{2k+1}, \cdots, i_{2m}\}$. We will obtain

$$a_{u_1, u_2, \cdots, u_{2r+1}, i_{2k+1}, \cdots, i_{2n}} \leq c(k, c, d) n^{-\frac{1}{2} - l_1 - l_4 - l_5 + 1}$$

with $c(k, c, d)$ defined as in (4.26). The sum over $u_1, \cdots, u_{r+1}$ from 1 to n reduces to a sum of $l_1 + l_3 + l_5 - 1$ distinct indices from 1 to n. Therefore we get

$$\begin{aligned} &\sum_{u_1, \cdots, u_{2r+1} = 1}^{n} a_{u_1, u_2, \cdots, u_{2r+1}, i_1, \cdots, i_{2k}} a_{u_1, u_2, \cdots, u_{2r+1}, i_{2k+1}, \cdots, i_{2n}} \\ &\leq c(k, r, m) n^{-l_1 - l_2 - l_4} \end{aligned}$$

with

$$c(k, r, m) = \sum_{x+y=2m} \sum_{c_1 + \cdots + c_{l_1} = x} \sum_{d_1 + \cdots + d_{l_2} = y} \sum_{e_1 + \cdots + e_{l_3} = r - x} c(r, c, e) c(k, c, d). \tag{4.27}$$

We need to consider the sum $i_1, \cdots, i_{2m}$ from 1 to n. It reduces to a sum over $l_2 + l_4$

distinct indices. Thus

$$\sum_{i_1,\cdots,i_{2m}=1}^{n}\left(\sum_{k=0}^{2m}\frac{1}{2k!}\frac{1}{(2m-2k)!}\sum_{r\geq 0}\frac{1}{(2r)!}\frac{1}{2m-2k+2r+1}\sum_{u_1,\cdots,u_{2r+1}=1}^{n}\sum_{u_1,\cdots,u_{2r+1}=1}^{n}\right.$$
$$\left. a_{u_1,u_2,\cdots,u_{2r+1},i_1,\cdots,i_{2k}}a_{u_1,u_2,\cdots,u_{2r+1},i_{2k+1},\cdots,i_{2m}}\right)^2$$
$$\leq n^{l_2+l_4}\left(\frac{1}{n^{2l_1+l_2+l_4}}\right)^2\sum_{k=0}^{2m}\frac{1}{2k!}\frac{1}{(2m-2k)!}\sum_{r\geq 0}\frac{1}{(2r)!}\frac{1}{2m-2k+2r+1}c(k,r,m)$$
$$=\frac{1}{n^{2l_1+l_2+l_4}}\left(\sum_{k=0}^{2m}\frac{1}{2k!}\frac{1}{(2m-2k)!}\sum_{r\geq 0}\frac{1}{(2r)!}\frac{1}{2m-2k+2r+1}c(k,r,m)\right)^2.$$

Note that either $l_1+l_2\geq 1$ or $l_1+l_4\geq 1$ (this is true because $m\geq 1$). Then this term is at most of order of n^{-1}.
Let us now look at the second part of the sum in (4.25). Suppose that in the sets $\{u_1,\cdots,u_{2r}\}\cap\{i_1,\cdots,i_{2k+1}\}$, $\{i_1,\cdots,i_{2k+1}\}\backslash\{u_1,\cdots,u_{2r}\}$, $\{u_1,\cdots,u_{2r}\}\backslash\{i_1,\cdots,i_{2k+1}\}$, $\{i_{2k+2},\cdots,i_{2m-2k}\}\setminus\{u_1,\cdots,u_{2r}\}$, $\{u_1,\cdots,u_{2r}\}\setminus\{i_{2k+2},\cdots,i_{2m-2k}\}$ we have sequences with lengths

$$p_1,\;\; p_2,\;\; p_3,\;\; p_4,\;\; p_5$$

respectively (the analogous of $l_1,\cdots,l_5$ above). Then the behavior with respect to n of

$$\sum_{u_1,\cdots,u_{2r}=1}^{n}a_{u_1,u_2,\cdots,u_{2r},i_1,\cdots,i_{2k+1}}a_{u_1,u_2,\cdots,u_{2r},i_{2k+2},\cdots,i_{2m}}$$

is of order of $n^{p_1+p_3}\frac{1}{n^{2p_1+p_3+p_4}}$. Therefore the behavior with respect to n of the second sum in (4.25) is of order

$$n^{p_2+1+p_4+1}\left(\frac{1}{n^1+2p_1+p_2+p_4}\right)^2=\frac{1}{n^{2p_1+p_2+p_4}}.$$

Again, since either $p_1+p_2\geq 1$ or $p_1+p_4\geq 1$, the behavior of the term is at most of order n^{-1}. Therefore

$$\mathbf{E}\left(\left(\langle DF_n, D(-L)^{-1}F_n\rangle-1\right)^2\right)\leq\frac{c_0}{n}$$

where the constant c_0 is given by (4.24). The fact that the sum over m is finite is a consequence of the following argument : $\langle DF_n, D(-L)^{-1}F_n\rangle$ belongs to $\mathbb{D}^{\infty,2}(\Omega)$ (which is true based on the derivation rule - Exercise 1.2.13 in [Nua06]- and since F_n belongs to $\mathbb{D}^{\infty,2}$ as a consequence of Proposition 1.2.3 in [Nua06]), this implies that $\sum_m m!m^k\|h_m^{(n)}\|_2<\infty$ for every k where $h_m^{(n)}$ is given by (4.23). Therefore, the constant $c(m,k,r)$ defined in (4.27) behaves at most as a power function with respect to m. ■

Corollary 1. *Let $J_m(F_n)$ denotes the projection on the m^{th} Wiener chaos of the random variable F_n. Then for every $m\geq 1$ the sequence $J_m(F_n)$ converges as $n\to\infty$ to a standard normal random variable.*

Proof : Note that if one considers the projection of F_n on the m^{th} Wiener chaos, one only deals with one of the terms of the chaotic decomposition of F_n. Because of this, the proof of Theorem 13 simplifies to dealing with one term instead of the sum of all the projections. Therefore, we immediatly obtain the convergence of each chaos to a standard normal random variable by mimicking the same proof in the case of the projection. ■

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Troisième partie

Regularity of the solutions to backward stochastic differential equations

Chapitre 5

Density estimates for solutions to one dimensional SDE's and Backward SDE's

Omar Aboura * and Solesne Bourguin †
Université Paris 1 and *Université Paris 1*

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Abstract

In this paper, we give sufficient conditions for the solutions of stochastic differential equations and backward stochastic differential equations to have a density for which we give upper and lower estimates. In the case of backward SDEs, the density estimates we derive are Gaussian.

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5.1 Introduction

In [NV09], I. Nourdin and F.G. Viens have introduced sufficient conditions to prove the existence of a density for a Malliavin differentiable random variable and to provide upper and lower Gaussian estimates for this density.

This result has lead to several research papers, such as those by D. Nualart and L. Quer-Sardanyons ([NQS09], [NQS11]), in which these authors applied Nourdin and Viens result to solutions of quasi-linear stochastic partial differential equations and to a class of stochastic equations with additive noise.

∗. SAMM, Université de Paris 1 Panthéon-Sorbonne, 90, rue de Tolbiac, 75634, Paris, France. Email : omar.aboura@malix.univ-paris1.fr

†. SAMM, Université de Paris 1 Panthéon-Sorbonne, 90, rue de Tolbiac, 75634, Paris, France. Email : solesne.bourguin@univ-paris1.fr

In this paper, we use Nourdin and Viens's approach to prove that, under proper conditions on the coefficients, each component of the solution (X_t, Y_t, Z_t) to a backward stochastic differential equation

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s & (5.1) \\ Y_t = \phi(X_T) + \int_t^T f(X_s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s & (5.2) \end{cases}$$

has a density for which upper and lower Gaussian bounds can be derived. This implies to study the diffusion equation (5.1) (which stands for itself) and provide upper and lower bounds for its density on one hand, and the backward SDE (5.2) on an other hand.

Our paper is organized in two main parts, the first one dealing with diffusions and the second one with backward SDEs. The question of the existence of a density for the solution to an SDE of the type (5.1) and the properties of this density has been intensively studied and we refer the reader to [Nua06] for an extensive survey of the existing litterature and results on this topic. See also [Nua04] for the case of non-linear SDEs.

We establish that under a sign condition on σ and a growth condition on the Lie bracket of b and σ (see Hypotheses (**H1**) and (**H2**)), (5.1) has a density for which upper and lower estimates can be derived. We also study the same question in the backward SDEs setting, where we consider equations of the type (5.2). These equations introduced in [PP90], which are closely related with viscosity solution to PDEs, have been intensively studied and have many applications in control theory and financial methods among others.

The existence of the density for the random variable Y_t at a fixed time $t \in (0, T)$, as well as upper bounds for its tail behavior, have been proven by F. Antonelli and A. Kohatsu-Higa [AKH05], using a different approach (Bouleau-Hirsch Theorem). We retrieve Antonelli and Kohatsu-Higa's existence result for the density of Y_t, and we also derive Gaussian estimates for it. In order to provide (additionally to the existence result itself) estimates for the density of Y_t, we need to slightly strengthen the hypotheses of Antonelli and Kohatsu-Higa.

We also address the question of the existence of a density for the random variable Z_t as well as the possibility of deriving Gaussian estimates for it. This question has not been solved in [AKH05]. We need the same hypotheses as in the case of Y_t, as well as additional ones, since Z_t can be expressed as a function of the Malliavin derivative of Y_t.

In order to be self contained, we at first give an overview of some elements of Malliavin calculus in Section 2, and the corresponding notations. In Section 3, we study the case of a diffusion and give sufficient conditions for the density of the solution to exist and admit upper and lower estimates (which need not be Gaussian, except in some particular cases). Section 4 is dedicated to the backward SDE case and is organized in two subsections, dealing respectively with the question of the existence of a density, as well as its Gaussian upper and lower estimates for Y_t and Z_t.

5.2 Framework, main tools and notations

5.2.1 Elements of Malliavin calculus

Consider the real separable Hilbert space $L^2([0,T])$ and $(W(\varphi), \varphi \in L^2([0,T]))$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{A}, P)$, that is a centered Gaussian family of random variables such that $\mathbf{E}(W(\varphi)W(\psi)) = \langle \varphi, \psi \rangle_{L^2([0,T])}$. For any integer $n \geq 1$, denote by I_n the multiple stochastic integral with respect to W (see [Nua06] for an extensive survey on Malliavin calculus). The map I_n is actually an isometry between the Hilbert space $L^2([0,T]^n)$ equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{L^2([0,T]^n)}$ and the Wiener chaos of order n, which is defined as the closed linear span of the random variables $H_n(W(\varphi))$ where $\varphi \in L^2([0,T]), \|\varphi\|_{L^2([0,T])} = 1$ and H_n is the Hermite polynomial of degree $n \geq 1$, that is defined by

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \qquad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as follows : for positive integers m, n,

$$\begin{aligned} \mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle f, g\rangle_{L^2([0,T]^n)} \quad \text{if } m = n, \\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0 \quad \text{if } m \neq n. \end{aligned}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n\geq 0} I_n(f_n) \tag{5.3}$$

where $f_n \in L^2([0,T]^n)$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

Let L be the Ornstein-Uhlenbeck operator defined by $LF = -\sum_{n\geq 0} nI_n(f_n)$ if F is given by (5.3). For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables (see (1.28) in [Nua06]) with respect to the norm defined by

$$\|F\|_{\alpha,p} = \|(I - L)^{\frac{\alpha}{2}} F\|_{L^p(\Omega)},$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(W(\varphi_1), \ldots, W(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in L^2([0,T])$) as follows :

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_n))\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}\left(L^2([0,T])\right)$. The adjoint of D is denoted by δ and is called the divergence (or Skorohod) integral. It is a continuous operator from $\mathbb{D}^{\alpha,p}\left(L^2([0,T])\right)$ into $\mathbb{D}^{\alpha+1,p}$. More generally, we can introduce iterated weak derivatives of order k. If F is a smooth random variable and k is a positive integer, we set

$$D^k_{t_1,\ldots,t_k} F = D_{t_1} D_{t_2} \ldots D_{t_k} F.$$

We have the following duality relationship between D and δ

$$\mathbf{E}(F\delta(u)) = \mathbf{E}\langle DF, u\rangle_{L^2([0,T])} \text{ for every smooth } F.$$

For adapted integrands, the divergence integral coincides with the classical Itô integral. We will use the notation

$$\delta(u) = \int_0^T u_s dW_s.$$

Note that the following integration by parts relationship between D and δ holds

$$D_t(\delta(u)) = u_t + \int_0^T D_t u_s dW_s,$$

with $u \in \mathbb{D}^{1,2}(L^2([0,T]))$ such that $\delta(u) \in \mathbb{D}^{1,2}$.

5.2.2 Density existence and Gaussian estimates

In [NV09], Corollary 3.5, Nourdin and Viens have given the following sufficient condition for a weakly differentiable random variable to have a density with lower and upper Gaussian estimates.

Proposition 3. *Let F be in $\mathbb{D}^{1,2}$ and let the function g be defined for all $x \in \mathbb{R}$ by*

$$g(x) = \mathbf{E}\left(\langle DF, -DL^{-1}F\rangle_{L^2([0,T])} \middle| F - \mathbf{E}(F) = x \right). \tag{5.4}$$

If there exist positive constants $\gamma_{\min}, \gamma_{\max}$ such that, for all $x \in \mathbb{R}$, almost surely

$$0 < \gamma_{\min}^2 \leqslant g(x) \leqslant \gamma_{\max}^2$$

then F has a density ρ satisfying, for almost all $z \in \mathbb{R}$

$$\frac{\mathbf{E}|F - \mathbf{E}(F)|}{2\,\gamma_{\max}^2} \exp\left(-\frac{(z - \mathbf{E}(F))^2}{2\gamma_{\min}^2} \right) \leqslant \rho(z) \leqslant \frac{\mathbf{E}|F - \mathbf{E}(F)|}{2\,\gamma_{\min}^2} \exp\left(-\frac{(z - \mathbf{E}(F))^2}{2\gamma_{\max}^2} \right).$$

Furthermore, Nourdin and Viens have also provided the following useful result, which gives some rather explicit description of $g(x)$. Recall that $W = (W(\phi), \phi \in L^2([0,T]))$.

Proposition 4. *Let F be in $\mathbb{D}^{1,2}$ and write $DF = \Phi_F(W)$ with a measurable function $\Phi_F : \mathbb{R}^{L^2([0,T])} \to L^2([0,T])$. Then, if $g(x)$ is defined by (5.4), we have*

$$g(x) = \int_0^\infty e^{-u}\, \mathbf{E}\left(\mathbf{E}'(\langle \Phi_F(W), \widetilde{\Phi_F^u}(W)\rangle_{L^2([0,T])}) | F - \mathbf{E}(F) = x \right) du,$$

where $\widetilde{\Phi_F^u}(W) = \Phi_F(e^{-u}W + \sqrt{1-e^{-2u}}W')$, W' stands for an independent copy of W, and is such that W and W' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P}')$ and $\mathbf{E}'$ denotes the mathematical expectation with respect to $\mathbb{P}'$.

5.2.3 Notations

Let f be a twice differentiable function of two variables x and y. We will use the following notations : $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2}(x,y) = f_{yy}$, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, $\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$. We will also use the following notation for the Lie bracket : $[f,g] = fg' - gf'$.

In the whole paper, c and C will denote constants that may vary from line to line.

5.3 Density estimates for one dimensional SDEs

Consider the following one dimensional stochastic differential equation

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{5.5}$$

where $x_0 \in \mathbb{R}$, b and σ are appropriately smooth functions to ensure the existence and uniqueness of solutions and $(W_t)_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$. In this section, we establish under what conditions the solution to (5.5) has a density for which upper and lower estimates can be derived. Expect for some particular cases such as the Ornstein-Uhlenbeck process, these estimates are not Gaussian and are defined on the support of the density. We will start by giving the hypotheses we will be working with before stating the main result of this section, i.e., the upper and lower estimates of the density of the solution to (5.5).

5.3.1 Hypotheses and examples

We consider b and σ to be C^2 Lipschitz functions, which ensures the existence and uniqueness of solutions to (5.5). In addition to that, we impose the following conditions :

$$\begin{cases} \textbf{H0 :} \begin{cases} \text{For every } t > 0,\ \sigma(x) > 0 \text{ a.e. on the support of } X_t \\ \text{Moreover we suppose that } supp(X_t) \text{ is an interval independent of } t > 0 \end{cases} \\ \textbf{H1 :} \ \exists\ M_l \geq 0, |\,[b,\sigma]\,| \leq M_l|\sigma| \\ \textbf{H2 :} \ \exists\ M_{\sigma\sigma''} \geq 0, |\sigma\sigma''| \leq M_{\sigma\sigma''} \end{cases} \tag{5.6}$$

where $[b,\sigma]$ denotes the Lie bracket of b and σ.

Remark 12. *Hypothesis (**H0**) on the positivity of σ on the support of X_t is not a loss of generality. In fact, we only need σ to keep the same sign on the support of X_t. The case where σ is negative was included neither in the proofs nor in the hypotheses for the sake of clarity and readability of the paper. However this case can be addressed (without any further difficulties) by using the following transformations : $\sigma \to \tilde{\sigma} := -\sigma$ and $W \to \tilde{W} := -W$. After performing those tranformations, it suffices to consider X to be the solution of*

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \tilde{\sigma}(X_s)d\tilde{W}_s.$$

This brings the problem back to the above set of hypotheses and it can be dealt with by the exact same arguments.

Here are some examples of coefficients b and σ satisfying hypotheses (**H0**) – (**H2**).

Example 2. *Consider the particular case where $(X_t)_{t\geq 0}$ is the drifted Brownian motion, i.e. $x_0 = 0$, $b(x) = b$ and $\sigma(x) = \sigma \neq 0$. It is clear that $[b,\sigma] = 0 = \sigma\sigma''$, $supp(X_t) = \mathbb{R}$, and that hypotheses (**H0**) – (**H2**) are satisfied.*

Example 3. *Consider the particular case where $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process, i.e. $b(x) = bx$, $b \in \mathbb{R}$ and $\sigma(x) = \sigma \neq 0$ Thus, $[\sigma, b] = b\sigma$, $\sigma\sigma'' = 0$, $supp(X_t) = \mathbb{R}$, and hypotheses (**H0**) – (**H2**) are satisfied.*

Example 4. *Consider the particular case where $(X_t)_{t\geq 0}$ is a geometric Brownian motion, i.e. $x_0 \neq 0$, $b(x) = bx$, $b \in \mathbb{R}$ and $\sigma(x) = \sigma x$, $\sigma \neq 0$ with $\sigma x_0 > 0$. Thus, $[b,\sigma] = 0 = \sigma\sigma''$, if $x_0 > 0$ then $supp(X_t) = [0,\infty)$ and we suppose that $\sigma > 0$ (resp. if $x_0 < 0$, then $supp(X_t) = (-\infty, 0]$ and we suppose that $\sigma < 0$). The hypotheses (**H0**) – (**H2**) are satisfied.*

5.3.2 Main result (Existence and estimates for the density of X)

The following result provides upper and lower estimates for the solutions to (5.5).

Theorem 14. *Consider equation (5.5) and let G be an antiderivative of $\frac{1}{\sigma}$. Under the hypotheses of Subsection 5.3.1, for $t \in (0,T]$ the random variable X_t has a density ρ_{X_t}. Furthermore, there exist strictly positive constants c and C such that, for almost all $x \in \mathbb{R}$, ρ_{X_t} satisfies the following :*

$$\rho_{X_t}(x) \geq \mathbf{1}_{supp(X_t)}(x) \frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)Ct} e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2ct}} \tag{5.7}$$

and

$$\rho_{X_t}(x) \leq \mathbf{1}_{supp(X_t)}(x) \frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)ct} e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2Ct}}. \tag{5.8}$$

Note that $G|_{supp(X_t)}$ is invertible and that $supp(X_t) = Im(\left\{G|_{supp(X_t)}\right\}^{-1})$ does not depend on the antiderivative G.

Remark 13. *Note that the support of the density ρ_{X_t} is not necesseraly $\mathbb{R}$, but $supp(X_t)$.*

Here are some examples of bounds derived on classical processes using Theorem 14.

Example 5. *Consider the particular case where $X_t = x_0 + \sigma W_t + bt$, i.e. $x_0 \in \mathbb{R}$, $b(x) = b$ and $\sigma(x) = \sigma$. We have $G(x) = \frac{x}{\sigma} + cst$ and $X_t(\Omega) = \mathbb{R}$. Thus the bounds (5.7) and (5.8) become*

$$\frac{1}{2C\sigma t} e^{-\frac{(x-bt-x_0)^2}{2c\sigma^2 t}} \leq \rho_{X_t}(x) \leq \frac{1}{2c\sigma t} e^{-\frac{(x-bt-x_0)^2}{2C\sigma^2 t}}$$

Example 6. *Consider the particular case where $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process, i.e. $b(x) = bx$, $b \in \mathbb{R}^*$ and $\sigma(x) = \sigma \in \mathbb{R}$. Then $X_t \sim \mathcal{N}\left(x_0 e^{bt}, \frac{\sigma^2}{2b}(e^{2bt-1})\right)$. We have $G(x) = \frac{x}{\sigma} + cst$ and $X_t(\Omega) = \mathbb{R}$. Thus the bounds (5.7) and (5.8) become*

$$\frac{\sqrt{e^{2bt}-1}}{2\sigma Ct\sqrt{b}} e^{-\frac{(x-x_0e^{bt})^2}{2\sigma^2 ct}} \leq \rho_{X_t}(x) \leq \frac{\sqrt{e^{2bt}-1}}{2\sigma ct\sqrt{b}} e^{-\frac{(x-x_0e^{bt})^2}{2\sigma^2 Ct}}$$

Example 7. *Consider the particular case where $(X_t)_{t\geq 0}$ is a geometric Brownian motion, i.e. $x_0 \neq 0$, $b(x) = bx$, $b \in \mathbb{R}$ and $\sigma(x) = \sigma x$, $\sigma \neq 0$ with $\sigma x_0 > 0$. We have, for $x \neq 0$, $G(x) = \frac{\ln(|x|)}{\sigma} + cst$ and if $x_0 > 0$, $X_t(\Omega) =]0, +\infty[$ (resp. $X_t(\Omega) =]-\infty, 0[$ if $x_0 < 0$). Thus the bounds (5.7) and (5.8) become*

$$\frac{\mathbf{1}_{supp(X_t)}(x)}{4\sigma xCt} e^{-\frac{\left(\ln(|x|)-\ln(|x_0|)-\left(b-\frac{\sigma^2}{2}\right)t\right)^2}{2c\sigma^2 t}} \leq \rho_{X_t}(x) \leq \frac{\mathbf{1}_{supp(X_t)}(x)}{4\sigma xct} e^{-\frac{\left(\ln(|x|)-\ln(|x_0|)-\left(b-\frac{\sigma^2}{2}\right)t\right)^2}{2C\sigma^2 t}}.$$

Let us first prove the following Lemma that will be useful for the proof of Theorem 14.

Lemma 20. *For every $T > 0$, $x \in \mathbb{R}^*$, there exist positive constants c and C such that for every $t \in [0,T]$,*

$$ct \leq \frac{e^{xt}-1}{x} \leq Ct.$$

Proof : If $x > 0$, Taylor's formula implies that for $t \in [0, T]$,

$$t \leq \frac{e^{xt} - 1}{x} = \int_0^t e^{xs} ds \leq t e^{xT}.$$

Similarly, if $x < 0$, we have, for $t \in [0, T]$, $te^{xT} \leq \frac{e^{xt}-1}{x} \leq t$. ■

Proof of Theorem 14 : Recall that G denotes an antiderivative of $\frac{1}{\sigma}$. Then G is strictly increasing on $supp(X_t)$ and we denote by G^{-1} the inverse map of $G : supp(X_t) \to G(supp(X_t))$. Let U_t be defined by

$$U_t = G(X_t) \Leftrightarrow X_t = G^{-1}(U_t). \tag{5.9}$$

Remark 14. *Note at first that G does not depend on t (as an antiderivative of $\frac{1}{\sigma}$), and that (**H0**) implies that the restriction of G to the support of X_t is invertible since $supp(X_t)$ is assumed to be an interval independent of t. The invertibility of G reduced to the interior of the support of X_t, $\overset{\circ}{supp}(X_t)$, is the only assumption that is required for the proof.*

Applying Itô's formula to $G(X_t)$ and using the identity $G'(x) = \frac{1}{\sigma(x)}$, we obtain

$$\begin{aligned}
dU_t =& G'(X_t) dX_t + \frac{1}{2} G''(X_t) d\langle X \rangle_t \\
=& \left[G'(X_t) b(X_t) + \frac{1}{2} G''(X_t) \sigma^2(X_t) \right] dt + G'(X_t)\sigma(X_t) dW_t \\
=& \beta \circ G^{-1}(U_t) dt + dW_t,
\end{aligned}$$

where β is defined by

$$\beta(x) = G'(x) b(x) + \frac{1}{2} G''(x)\sigma^2(x) = \frac{b}{\sigma}(x) - \frac{\sigma'(x)}{2}. \tag{5.10}$$

Thus,

$$U_t = G(x_0) + \int_0^t \beta \circ G^{-1}(U_s) ds + W_t$$

and for $\theta \in [0, t]$ we have

$$D_\theta U_t = 1 + \int_\theta^t (\beta \circ G^{-1})'(U_s) D_\theta U_s ds = \exp\left[\int_\theta^t (\beta \circ G^{-1})'(U_s) ds \right]. \tag{5.11}$$

Deriving the identity $G \circ G^{-1}(x) = x$ on $G(supp(X_t))$ yields $(G^{-1})'(x) = \sigma \circ G^{-1}(x)$. Using this fact we get $(\beta \circ G^{-1})'(x) = \beta' \circ G^{-1}(x)(G^{-1})'(x) = (\beta'\sigma) \circ G^{-1}(x)$. In addition, it is easy to check that on $G(supp(X_t))$,

$$(\beta'\sigma)(x) = \frac{[\sigma, b](x)}{\sigma(x)} - \frac{(\sigma\sigma'')(x)}{2}. \tag{5.12}$$

Gathering those results and using hypotheses (**H1**) and (**H2**) of Subsection 5.3.1 immediatly yields on $G(supp(X_t))$

$$-\left(M_l + \frac{M_{\sigma\sigma''}}{2} \right) \leq (\beta \circ G^{-1})' \leq \left(M_l + \frac{M_{\sigma\sigma''}}{2} \right).$$

Using (5.11), we deduce, $\mathbb{P}$-a.s,

$$0 < e^{-\left(M_l+\frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)} \leq D_\theta U_t \leq e^{\left(M_l+\frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)}. \tag{5.13}$$

Write $D_\bullet U_t = \Phi_{U_t}^\bullet(W)$ with a measurable function $\Phi_{U_t}^\bullet : \mathbb{R}^{L^2([0,T])} \longrightarrow L^2([0,T])$. Then (5.13) becomes, for $\theta < t$,

$$0 < e^{-\left(M_l+\frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)} \leq \Phi_{U_t}^\theta(W) \leq e^{\left(M_l+\frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)}. \tag{5.14}$$

Define $\widetilde{\Phi_{U_t}^{\bullet,u}}(W) = \Phi_{U_t}^\bullet(e^{-u}W + \sqrt{1-e^{-2u}}W')$ for $u \in [0,+\infty[$, where W' stands for an independent copy of W and is such that W and W' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P}')$. Using (5.13), it is clear that the following holds, for $\theta < t$, $u \in [0, \infty)$

$$0 < e^{-\left(M_l+\frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)} \leq \widetilde{\Phi_{U_t}^{\theta,u}}(W) \leq e^{\left(M_l+\frac{M_{\sigma\sigma''}}{2}\right)(t-\theta)}. \tag{5.15}$$

Using Proposition 4 for $g(x) = \mathbf{E}\left(\langle DU_t, -DL^{-1}U_t\rangle_{L^2([0,T])}\Big|U_t - \mathbf{E}(U_t) = x\right)$, we have

$$\begin{aligned}
g(x) &= \int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'(\langle\Phi_{U_t}^\bullet(W), \widetilde{\Phi_{U_t}^{\bullet,u}}(W)\rangle_{L^2([0,T])})|U_t - \mathbf{E}(U_t) = x\right) du \\
&= \int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'\left(\int_0^t \Phi_{U_t}^\theta(W)\widetilde{\Phi_{U_t}^{\theta,u}}(W)d\theta\right)|U_t - \mathbf{E}(U_t) = x\right) du.
\end{aligned}$$

Using the bounds in (5.14) and (5.15), we obtain, $\mathbb{P}$-a.s,

$$0 < \int_0^\infty e^{-u}\int_0^t e^{-(2M_l+M_{\sigma\sigma''})(t-\theta)}d\theta du \leq g(x) \leq \int_0^\infty e^{-u}\int_0^t e^{(2M_l+M_{\sigma\sigma''})(t-\theta)}d\theta du,$$

which leads to, $\mathbb{P}$-a.s,

$$0 < \frac{1-e^{-(2M_l+M_{\sigma\sigma''})t}}{2M_l+M_{\sigma\sigma''}} \leq g(x) \leq \frac{e^{(2M_l+M_{\sigma\sigma''})t}-1}{2M_l+M_{\sigma\sigma''}}.$$

Lemma 20 implies the existence of strictly positive constants c and C such that, for $t \in (0,T]$,

$$0 < ct \leq g(x) \leq Ct \quad \mathbb{P}-a.s.$$

Using (5.13) we deduce that $U_t \in \mathbb{D}^{1,2}$. Hence Proposition 3 implies that U_t has a density ρ_{U_t} and that there exist constants c and C such that $0 < c < C$ and for $u \in G(supp(X_t))$,

$$\frac{\mathbf{E}|U_t - \mathbf{E}(U_t)|}{2Ct}\exp\left(-\frac{(u-\mathbf{E}(U_t))^2}{2ct}\right) \leq \rho_{U_t}(u) \leq \frac{\mathbf{E}|U_t - \mathbf{E}(U_t)|}{2ct}\exp\left(-\frac{(u-\mathbf{E}(U_t))^2}{2Ct}\right).$$

We now prove that for any $t \in (0,T]$, X_t has a density, which we compare to that of U_t. For any bounded Borel function f, (using the change of variable $x = G^{-1}(u)$) for all $x \in supp(X_t)$, we deduce

$$\begin{aligned}
\mathbf{E}\left(f(X_t)\right) =& \mathbf{E}\left(f \circ G^{-1}(U_t)\right) = \int_{G(supp(X_t))} f \circ G^{-1}(u)\rho_{U_t}(u)du \\
=& \int_{supp(X_t)} f(x)\frac{\rho_{U_t} \circ G(x)}{\sigma(x)}dx.
\end{aligned}$$

Using this, we can recover that X_t has a density ρ_{X_t} such that

$$\rho_{X_t}(x) = \frac{\rho_{U_t} \circ G(x)}{\sigma(x)} \mathbf{1}_{supp(X_t)}(x).$$

Hence, the upper and lower estimates of ρ_{U_t} yield

$$\rho_{X_t}(x) \geq \mathbf{1}_{supp(X_t)}(x) \frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)Ct} e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2ct}}$$

and

$$\rho_{X_t}(x) \leq \mathbf{1}_{supp(X_t)}(x) \frac{\mathbf{E}|G(X_t) - \mathbf{E}(G(X_t))|}{2\sigma(x)ct} e^{-\frac{(G(x)-\mathbf{E}(G(X_t)))^2}{2Ct}}.$$

This concludes the proof of Theorem 14. ■

5.4 Gaussian density estimates for one dimensional backward SDEs

5.4.1 Preliminaries

The following backward stochastic differential equation was introduced in Pardoux and Peng [PP90] (see also [PP92]) and was also studied in [AKH05] :

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \\ Y_t = \phi(X_T) + \int_t^T f(X_s, Y_s, Z_s)\,ds - \int_t^T Z_s dW_s. \end{cases} \tag{5.16}$$

In this section, we give conditions for the random variables Y_t and Z_t to have a density which can be bounded from above and below by Gaussian ones. We will first focus on the case of Y_t for which we have to impose strict ellipticity conditions on the coefficients of X along with some additional hypotheses on the coefficients of Y. The case of Z_t requires stronger assumptions on the coefficients of X and Y that will be detailed in the dedicated subsection. Note that, as opposed to the case of the diffusion process X_t, the estimates we obtain for the backward part (Y, Z) of equation (5.16) are always Gaussian.

5.4.2 Notations

We introduce here some notations in use in this section. Let $B_0^{n,+}(\mathbb{R})$ be the space of bounded $\mathcal{C}^n(\mathbb{R})$ functions which are bounded positively away from 0 such that their derivatives up to order n are bounded, i.e., $f \in B_0^{n,+}(\mathbb{R})$ if and only if there exist positive constants c, C and $M_{f^{(i)}}$, $i = 1, ..., n$, such that $0 < c \leq f \leq C$ and for each $i = 1, ..., n$, $\left|f^{(i)}\right| \leq M_{f^{(i)}}$. Let $B_0^{n,-}(\mathbb{R})$ be the space of $\mathcal{C}^n(\mathbb{R})$ functions such that $-f \in B_0^{n,+}(\mathbb{R})$ and $B_0^n(\mathbb{R}) = B_0^{n,+}(\mathbb{R}) \cup B_0^{n,-}(\mathbb{R})$.

5.4.3 Density of Y_t : existence and Gaussian estimates

We first give the set of hypotheses (in terms of the diffusion's coefficients and the backward equation's coefficients) that will be needed in the main theorem on the density of Y_t.

Hypotheses

Consider equation (5.16). On the diffusion part, we still consider b and σ to be appropriately smooth functions to ensure the existence and uniqueness of solutions to the first equation in (5.16). We also need to impose these two additional conditions on b and σ :

$$\begin{cases} \mathbf{H3:} \ \exists\, M_l \geq 0, \quad |[b,\sigma]| \leq M_l \\ \mathbf{H4:} \ \sigma \in B_0^{2,+}(\mathbb{R}) \end{cases}$$

where $[b,\sigma]$ denotes the Lie bracket between b and σ.

Remark 15. *Recall that $\sigma \in B_0^{2,+}(\mathbb{R})$ implies that there exist two strictly positive constants that will be refered to as m_σ and M_σ such that $0 < m_\sigma \leq \sigma \leq M_\sigma$.*

Hence if $\sigma \in B_0^{2,+}(\mathbb{R})$, the assumption (**H1**) is equivalent to $|[b,\sigma]| \leq M$ for some positive constant M. Clearly, (**H0**) and (**H2**) are also satisfied if $\sigma \in B_0^{2,+}(\mathbb{R})$.
On the backward part of (5.16), we make the following assumptions :

$$\begin{cases} \mathbf{H5:} \ \exists\, c_{\phi'}, C_{\phi'}, \quad 0 < c_{\phi'} \leq |\phi'| \leq C_{\phi'} \\ \mathbf{H6:} \ \exists\, c_{f_x}, C_{f_x}, M_{f_y}, M_{f_z}, \quad \left\{ \begin{array}{l} 0 < c_{f_x} \leq |f_x| \leq C_{f_x} \\ |f_y| \leq M_{f_y} \quad |f_z| \leq M_{f_z} \end{array} \right\} \\ \mathbf{H7:} \ \forall u, v, \quad \phi'(u) f_x(v) > 0 \end{cases}$$

Main result (Existence and estimates for the density of Y_t)

Under the above assumptions, we have the following Gaussian estimates for the density of Y_t.

Theorem 15. *Under the hypotheses of Subsection 5.4.3, for $t \in (0,T)$ the random variable Y_t defined in (5.16) has a density ρ_{Y_t}. Furthermore, there exist strictly positive constants c and C such that, for almost all $y \in \mathbb{R}$, ρ_{Y_t} satisfies the following :*

$$\frac{\mathbf{E}|Y_t - \mathbf{E}(Y_t)|}{2ct} \exp\left(-\frac{(y-\mathbf{E}(Y_t))^2}{2Ct}\right) \leq \rho_{Y_t}(y) \leq \frac{\mathbf{E}|Y_t - \mathbf{E}(Y_t)|}{2Ct} \exp\left(-\frac{(y-\mathbf{E}(Y_t))^2}{2ct}\right).$$

Before proving Theorem 15, we will first prove the following Proposition that will play a key role in the upcoming proof of this Theorem.

Proposition 5. *Suppose that the conditions (**H3**) – (**H7**) hold. Then for $0 < \theta < t \leq T$, there exist some strictly positive constants $k_{Y,1}(\theta,t), k_{Y,2}(\theta,t)$ such that $\mathbb{P}$-a.s,*

$$0 < k_{Y,1}(\theta,t) \leq |D_\theta Y_t| \leq k_{Y,2}(\theta,t) \tag{5.17}$$

with

$$\begin{aligned} k_{Y,1}(\theta,t) =& c_{\phi'} m_\sigma e^{m_{b,\sigma}(T-\theta) - M_{f_y}(T-t)} \\ &+ \frac{c_{f_x} m_\sigma e^{M_{f_y} t - m_{b,\sigma}\theta}}{m_{b,\sigma} - M_{f_y}} \left(e^{(m_{b,\sigma} - M_{f_y})T} - e^{(m_{b,\sigma} - M_{f_y})t}\right) \end{aligned}$$

and

$$\begin{aligned} k_{Y,2}(\theta,t) =& C_{\phi'} M_\sigma e^{M_{b,\sigma}(T-\theta) + M_{f_y}(T-t)} \\ &+ \frac{C_{f_x} M_\sigma e^{-M_{f_y} t - M_{b,\sigma}\theta}}{M_{b,\sigma} + M_{f_y}} \left(e^{(M_{b,\sigma} + M_{f_y})T} - e^{(M_{b,\sigma} + M_{f_y})t}\right), \end{aligned}$$

where $m_{b,\sigma}$ and $M_{b,\sigma}$ are constants depending only on b and σ.

Proof : We at first represent $D_\theta Y_t$ by means of an equivalent probability ; this is similar to [AKH05] and the proof is included for the sake of completeness. It is well known (see for example Theorem 2.2 in [AKH05]) that, for every $t \in (0,T]$, $Y_t \in \mathbb{D}^{1,2}$ and $Z \in L^2\left(0,T;\mathbb{D}^{1,2}\right)$. Furthermore, since $\theta < t$, we have

$$\begin{aligned} D_\theta Y_t =& \phi'\left(X_T\right) D_\theta X_T - \int_t^T D_\theta Z_s dW_s \\ &+ \int_t^T \left[f_x\left(X_s,Y_s,Z_s\right) D_\theta X_s + f_y\left(X_s,Y_s,Z_s\right) D_\theta Y_s + f_z\left(X_s,Y_s,Z_s\right) D_\theta Z_s\right] ds. \end{aligned} \tag{5.18}$$

The product $e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} D_\theta Y_t$ yields a more suitable representation of $D_\theta Y_t$; indeed, for $t \in (0,T]$, and $0 \le \theta < t$

$$\begin{aligned} d\left[e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} D_\theta Y_t\right] =& \Big[D_\theta Y_t e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} f_y\left(X_t,Y_t,Z_t\right) \\ & - e^{\int_0^t f_y(X_s,Y_s,Z_s)ds}\left(f_x\left(X_t,Y_t,Z_t\right) D_\theta X_t + f_y\left(X_t,Y_t,Z_t\right) D_\theta Y_t\right. \\ & \left. + f_z\left(X_t,Y_t,Z_t\right) D_\theta Z_t\right)\Big] dt + e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} D_\theta Z_t dW_t. \end{aligned}$$

Integrating from t to T yields, for $\theta < t$,

$$\begin{aligned} e^{\int_0^T f_y(X_s,Y_s,Z_s)ds} D_\theta Y_T - e^{\int_0^t f_y(X_s,Y_s,Z_s)ds} D_\theta Y_t =& -\int_t^T e^{\int_0^s f_y(X_r,Y_r,Z_r)dr}\left[f_x\left(X_s,Y_s,Z_s\right) D_\theta X_s\right. \\ & \left. + f_z\left(X_s,Y_s,Z_s\right) D_\theta Z_s\right] ds + \int_t^T e^{\int_0^s f_y(X_r,Y_r,Z_r)dr} D_\theta Z_s dW_s. \end{aligned}$$

Note that $D_\theta Y_T = \phi'\left(X_T\right) D_\theta X_T$; therefore, for $t \in (0,T]$,

$$\begin{aligned} D_\theta Y_t =& e^{\int_t^T f_y(X_s,Y_s,Z_s)ds} \phi'\left(X_T\right) D_\theta X_T + \int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr}\left[f_x\left(X_s,Y_s,Z_s\right) D_\theta X_s\right. \\ & \left. + f_z\left(X_s,Y_s,Z_s\right) D_\theta Z_s\right] ds - \int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr} D_\theta Z_s dW_s. \end{aligned}$$

Let $\widetilde{W}_t = W_t - \int_0^t f_z\left(X_s,Y_s,Z_s\right) ds$. Because $|f_z| \le M_{f_z}$, Novikov's condition is verified and $\widetilde{W}$ is a Brownian motion under some equivalent probability $\widetilde{\mathbb{P}}$. Girsanov's theorem yields

$$\begin{aligned} D_\theta Y_t =& e^{\int_t^T f_y(X_s,Y_s,Z_s)ds} \phi'\left(X_T\right) D_\theta X_T + \int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr} f_x\left(X_s,Y_s,Z_s\right) D_\theta X_s ds \\ & - \int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr} D_\theta Z_s d\widetilde{W}_s. \end{aligned}$$

Conditionning by $\mathcal{F}_t$ under $\widetilde{\mathbb{P}}$ and taking into account the fact that Y_t and $D_\theta Y_t$ are adapted with respect to $\mathcal{F}_t$ while $\int_t^s f_y\left(X_r,Y_r,Z_r\right) dr$ and $D_\theta Z_s$ are $\mathcal{F}_s$-adapted for $\theta < t \le s \le T$, we obtain

$$\begin{aligned} D_\theta Y_t =& \widetilde{\mathbf{E}}\left(e^{\int_t^T f_y(X_s,Y_s,Z_s)ds} \phi'\left(X_T\right) D_\theta X_T \,\middle|\, \mathcal{F}_t\right) \\ & + \widetilde{\mathbf{E}}\left(\int_t^T e^{\int_t^s f_y(X_r,Y_r,Z_r)dr} f_x\left(X_s,Y_s,Z_s\right) D_\theta X_s ds \,\middle|\, \mathcal{F}_t\right). \end{aligned} \tag{5.19}$$

At this point, we need to make use of the random variable U_t introduced in the proof of Theorem 14. Recall that $U_t = G(X_t)$ where G is an antiderivative of $\frac{1}{\sigma}$. We have established in (5.11) that if G^{-1} is the inverse function of G restricted to $supp(X_t)$,

$$D_\theta U_t = \exp\left[\int_\theta^t (\beta \circ G^{-1})'(U_s)ds\right], \tag{5.20}$$

where β is defined by (5.10). Finally, recall that $(\beta \circ G^{-1})' = (\beta'\sigma) \circ G^{-1}$ where $\beta'\sigma$ is given by (5.12). Using hypotheses (**H3**) and (**H4**) of Subsection 5.4.3 as well as (5.13), we deduce the existence of two constants $m_{b,\sigma}$ and $M_{b,\sigma}$ such that for $0 < \theta < t \leq T$,

$$0 < e^{m_{b,\sigma}(t-\theta)} \leq D_\theta U_t \leq e^{M_{b,\sigma}(t-\theta)}. \tag{5.21}$$

Futhermore, as $X_t = G^{-1}(U_t)$, it holds that, for $0 < \theta < t \leq T$,

$$D_\theta X_t = (G^{-1})'(U_t) D_\theta U_t = \sigma \circ G^{-1}(U_t) D_\theta U_t. \tag{5.22}$$

Combining (5.21) and (5.22) with the fact that $0 < m_\sigma \leq \sigma \leq M_\sigma$ yields, $\mathbb{P}$-a.s (and $\widetilde{\mathbb{P}}$-a.s since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent),

$$0 < m_\sigma e^{m_{b,\sigma}(t-\theta)} \leq D_\theta X_t \leq M_\sigma e^{M_{b,\sigma}(t-\theta)}. \tag{5.23}$$

For every $t \in [0,T)$, $D_\theta X_t$ is positive and using (**H7**), we deduce that $\phi'(X_T)$ and $f_x(X_s, Y_s, Z_s)$ have the same sign. Hence both terms in the right hand side of (5.19) have the same sign. Using this fact along with hypotheses (**H5**), (**H6**) we estimate both terms in the right hand side of (5.19) and hence their sum ; this yields $\widetilde{\mathbb{P}}$-a.s for $0 < \theta < t \leq T$,

$$0 < k_{Y,1}(\theta, t) \leq |D_\theta Y_t| \leq k_{Y,2}(\theta, t) \tag{5.24}$$

with

$$\begin{aligned} k_{Y,1}(\theta,t) =& c_{\phi'} m_\sigma e^{m_{b,\sigma}(T-\theta) - M_{f_y}(T-t)} \\ &+ \frac{c_{f_x} m_\sigma e^{M_{f_y} t - m_{b,\sigma}\theta}}{m_{b,\sigma} - M_{f_y}} \left(e^{(m_{b,\sigma} - M_{f_y})T} - e^{(m_{b,\sigma} - M_{f_y})t}\right) \end{aligned}$$

and

$$\begin{aligned} k_{Y,2}(\theta,t) =& C_{\phi'} M_\sigma e^{M_{b,\sigma}(T-\theta) + M_{f_y}(T-t)} \\ &+ \frac{C_{f_x} M_\sigma e^{-M_{f_y} t - M_{b,\sigma}\theta}}{M_{b,\sigma} + M_{f_y}} \left(e^{(M_{b,\sigma} + M_{f_y})T} - e^{(M_{b,\sigma} + M_{f_y})t}\right). \end{aligned}$$

This concludes the proof of Proposition 5. ■

Remark 16. *In order to prove this proposition, we could also have considered a second BSDE that results from replacing the Malliavin derivatives and the derivatives of f that appear in (5.18) by their upper and lower estimates (given as hypotheses or computed in the previous section). After a similar Girsanov transform, we would then have used the comparison theorem for BSDEs (see for instance [Pen92] for statement and proof of this theorem) in order to obtain upper and lower bounds for the Malliavin derivative of Y.*

We are now ready to prove Theorem 15.

Proof of Theorem 15 : Write $D_\bullet Y_t = \Phi^\bullet_{Y_t}(W)$ with a measurable function $\Phi^\bullet_{Y_t}$: $\mathbb{R}^{L^2([0,T])} \longrightarrow L^2([0,T])$. Then Proposition 5 yields, for $\theta < t$,

$$0 < k_{Y,1}(\theta,t) \leq \left|\Phi^\theta_{Y_t}(W)\right| \leq k_{Y,2}(\theta,t). \tag{5.25}$$

Define $\widetilde{\Phi^{\bullet,u}_{Y_t}}(W) = \Phi^\bullet_{Y_t}(e^{-u}W + \sqrt{1-e^{-2u}}W')$ for $u \in [0,+\infty[$, where W' stands for an independent copy of W and is such that W and W' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P}')$. Using Proposition 5, it is clear that, for $\theta < t$, we have for any $u \in [0,\infty)$, $0 < k_{Y,1}(\theta,t) \leq \left|\widetilde{\Phi^{\theta,u}_{Y_t}}(W)\right| \leq k_{Y,2}(\theta,t)$. Noticing that $\Phi^\theta_{Y_t}(W)$ and $\widetilde{\Phi^{\theta,u}_{Y_t}}(W)$ have the same sign and combining the two previous bounds yields, for $\theta < t$, $u \in [0,\infty)$,

$$0 < k^2_{Y,1}(\theta,t) \leq \Phi^\theta_{Y_t}(W)\widetilde{\Phi^{\theta,u}_{Y_t}}(W) \leq k^2_{Y,2}(\theta,t). \tag{5.26}$$

Using the notation from Propositions 3 and 4,

$$g(y) = \int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'\left(\int_0^t \Phi^\theta_{Y_t}(W)\widetilde{\Phi^{\theta,u}_{Y_t}}(W)d\theta\right)\Big|Y_t - \mathbf{E}(Y_t) = y\right)du.$$

The bounds obtained in (5.26) immediatly yield

$$0 < \int_0^t k^2_{Y,1}(\theta,t)d\theta \leq g(y) \leq \int_0^t k^2_{Y,2}(\theta,t)d\theta. \tag{5.27}$$

We will now give lower (resp. upper) estimates of $A_1 = \int_0^t k^2_{Y,1}(\theta,t)d\theta$ (resp. $A_2 = \int_0^t k^2_{Y,2}(\theta,t)d\theta$). The constants c and C appearing in the calculations may change from line to line. We start by calculating a lower bound of A_1. Since both summands in $k_{Y,1}$ are positive, we have

$$k^2_{Y,1}(\theta,t) \geq ce^{2m_{b,\sigma}(T-\theta)-2M_{f_y}(T-t)}.$$

Thus,

$$A_1 \geq ce^{-2M_{f_y}(T-t)}\left[\frac{e^{2m_{b,\sigma}T} - e^{2m_{b,\sigma}(T-t)}}{2m_{b,\sigma}}\right] \geq \frac{e^{(2m_{b,\sigma}-2M_{f_y})(T-t)}}{2m_{b,\sigma}}(2m_{b,\sigma}ct),$$

where we used the fact that $e^y(x-y) \leq e^x - e^y$ if $x \geq y$. Because $e^{(2m_{b,\sigma}-2M_{f_y})(T-t)}$ is lower bounded by $e^{-2|M_{f_y}-m_{b,\sigma}|T}$, we finally obtain for some constant c depending on b, σ, f_y and T,

$$A_1 = \int_0^t k^2_{Y,1}(\theta,t)d\theta \geq ct. \tag{5.28}$$

It remains to prove that $A_2 \leq Ct$ for some constant C. Using Young's lemma, we can write

$$\begin{aligned} A_2 &\leq Ce^{2M_{f_y}(T-t)}\left[\frac{e^{2M_{b,\sigma}T} - e^{2M_{b,\sigma}(T-t)}}{2M_{b,\sigma}}\right] + Ce^{2(M_{b,\sigma}+M_{f_y})T}\int_0^t e^{-2M_{b,\sigma}\theta-2M_{f_y}t}d\theta \\ &\leq \frac{e^{2M_{f_y}(T-t)}}{2M_{b,\sigma}}e^{2M_{b,\sigma}T}(2M_{b,\sigma}Ct) + C\left[\frac{e^{-2M_{f_y}t} - e^{-2(M_{b,\sigma}+M_{f_y})t}}{2M_{b,\sigma}}\right]. \end{aligned}$$

This upper estimate and the fact that $e^x - e^y \le e^x(x-y)$ for $x \ge y$ yields

$$A_2 = \int_0^t k_{Y,2}^2(\theta, t)d\theta \le Ct. \tag{5.29}$$

The inequalities (5.27) – (5.29) yield, $\mathbb{P}$-a.s,

$$0 < ct \le g(y) \le Ct,$$

with strictly positive constants c and C. Thus, Propositions 3 and 4 conclude the proof of Theorem 15. ■

5.4.4 Density of Z_t : existence and Gaussian estimates

In this subsection, we will prove that under some conditions on the coefficients, Z_t has a density with Gaussian upper and lower bounds. We begin by listing those conditions in the upcoming subsection.

Hypotheses

We need to make additional assumptions with respect to those in subsections 5.3.1 and 5.4.3. More precisely, we assume (**H1**) and that the following holds on the diffusion process X_t,

$$\begin{cases} \textbf{H8 :}\ \sigma \in B_0^{3,+}(\mathbb{R}), \quad \sigma' \ge 0. \\ \textbf{H9 :}\ \exists\ M_l, M_{dl} \ge 0, \quad |[b,\sigma]| \le M_l\sigma, \quad 0 \le [\sigma,[\sigma,b]] \le M_{dl}\sigma. \end{cases}$$

where $[\phi, \psi]$ denotes the Lie bracket between ϕ and ψ.

Remark 17. *Recall that $\sigma \in B_0^{3,+}(\mathbb{R})$ implies that there exist strictly positive constants that will be refered to as m_σ and M_σ such that $0 < m_\sigma \le \sigma \le M_\sigma$. It also implies, along with the fact that $\sigma' \ge 0$, that there exist strictly positive constants that will be refered to as $M_{\sigma'}$, $M_{\sigma''}$ and $M_{\sigma^{(3)}}$ such that $0 \le \sigma' \le M_{\sigma'}$, $|\sigma''| \le M_{\sigma''}$ and $|\sigma^{(3)}| \le M_{\sigma^{(3)}}$.*

On the backward process (Y, Z), we need the following conditions on the functions ϕ and f, where $f : \mathbb{R}^2 \to \mathbb{R}$ does not depend on z :

$$\begin{cases} \textbf{H10 :}\ \text{There exist constants } c_{\phi'}, C_{\phi'}, C_{\phi''} \text{ such that } 0 < c_{\phi'} \le \phi' \le C_{\phi'},\ 0 < c_{\phi''} \le \phi'' \le C_{\phi''} \\ \textbf{H11 :}\ \text{There exist constants } m_{f_x}, M_{f_x}, M_{f_y}, M_{f_{xx}}, M_{f_{xy}}, M_{f_{yx}}, M_{f_{yy}} \text{ such that} \\ \qquad 0 < m_{f_x} \le f_x \le M_{f_x}, |f_y| \le M_{f_y}, 0 \le f_{xx} \le M_{f_{xx}}, 0 \le f_{xy} \le M_{f_{xy}}, 0 \le f_{yy} \le M_{f_{yy}} \end{cases}$$

Note that (**H10**) and (**H11**) imply (**H5**)-(**H7**).

Main result (Existence and estimates for the density of Z_t)

We consider equation (5.16) with a function $f^\star$ that only has a linear dependency on Z, i.e.

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \\ Y_t = \phi(X_T) + \int_t^T f^\star(X_s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s \end{cases}$$

where $f^\star(x, y, z) = f(x, y) + \alpha z$, $\alpha \in \mathbb{R}$.

Remark 18. *Because of the dependency of f on Z, the Malliavin derivative DZ will depend on D^2Z, which is not suitable for analyzing it within our framework. One can circumvent the above mentioned issue by using the Girsanov theorem to dispose of the impeding terms (similarly as done in the proof of Propositon 5). In order to clarify the proofs and to improve readability, we will consider that this step has already been performed in all of our proofs. This procedure leaves us with an equation of the type*

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \\ Y_t = \phi(X_T) + \int_t^T f(X_s, Y_s)\, ds - \int_t^T Z_s dW_s, \end{cases}$$

which is the one that will be referred to in the proofs of the upcoming results.

The following theorem provides Gaussian estimates for the density of Z_t.

Theorem 16. *Under the hypotheses of Subsection 5.4.4, for $t \in (0,T)$ the random variable Z_t has a density ρ_{Z_t}. Furthermore, there exist strictly positive constants c and C such that, for almost all $z \in \mathbb{R}$, ρ_{Z_t} satisfies the following :*

$$\frac{\mathbf{E}|Z_t - \mathbf{E}(Z_t)|}{2ct} \exp\left(-\frac{(z-\mathbf{E}(Z_t))^2}{2Ct}\right) \leq \rho_{Z_t}(z) \leq \frac{\mathbf{E}|Z_t - \mathbf{E}(Z_t)|}{2Ct} \exp\left(-\frac{(z-\mathbf{E}(Z_t))^2}{2ct}\right).$$

Before proving Theorem 16, we will first give a technical Lemma and a Proposition which will play a key role in the upcoming proof of this Theorem. First recall a lemma used to calculate the Malliavin derivative of a product of random variables in $\mathbb{D}^{1,2}$ (for example, see [Nua06], p.36, exercice 1.2.12).

Lemma 21. *(i) Let $s,t \in [0,T]$ and $F \in \mathbb{D}^{1,2}$; then we have $\mathbf{E}(F|\mathcal{F}_t) \in \mathbb{D}^{1,2}$ and*

$$D_s\mathbf{E}(F|\mathcal{F}_t) = \mathbf{E}(D_sF|\mathcal{F}_t)\, 1_{s\leq t}.$$

(ii) If $F, G \in \mathbb{D}^{1,2}$ are such that F and $\|DF\|_{L^2([0,T])}$ are bounded, then $FG \in \mathbb{D}^{1,2}$ and

$$D(FG) = FDG + GDF.$$

The following Proposition ensures that under the hypotheses of Subsection 5.4.4, the second order Malliavin derivatives of X and Y are positive and bounded from above.

Proposition 6. *Under the assumptions of section 5.4.4, there exist two positive constants M_{D^2X} and M_{D^2Y} such that for $0 < \theta < t < s \leq T$, $\mathbb{P}$-a.s,*

$$0 \leq D^2_{\theta,t}X_s \leq M_{D^2X} \quad \text{and} \quad 0 \leq D^2_{\theta,t}Y_s \leq M_{D^2Y}.$$

Remark 19. *Here we obtain large inequalities since the basic example of standard Brownian motion shows that the second Malliavin derivative of X_t may be null.*

Proof : We start by proving the inequalities on $D^2_{\theta,t}X_s$. Applying the Malliavin derivative to (5.22) and using the second point in Lemma 21, we deduce for $\theta, t \leq s \leq T$, since $U_s = G(X_s)$,

$$\begin{aligned} D^2_{\theta,t}X_s =&(\sigma \circ G^{-1})'(U_s)D_\theta U_s D_t U_s + (\sigma \circ G^{-1})(U_s)D^2_{\theta,t}U_s \\ =&(\sigma'\sigma)(X_s)D_\theta U_s D_t U_s + \sigma(X_s)D^2_{\theta,t}U_s. \end{aligned} \tag{5.30}$$

The hypotheses (**H8**) and (**H9**) along with (5.13) ensure that the term $(\sigma'\sigma)(X_s)D_\theta U_s D_t U_s$ is non negative and can be bounded from above by a constant given by

$$0 \leq (\sigma'\sigma)(X_s)D_\theta U_s D_t U_s \leq M_{\sigma'} M_\sigma e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(2s-t-\theta)}. \tag{5.31}$$

It remains to prove that the second summand in (5.30) is also non negative and bounded from above. As σ is non negative and bounded, we focus on proving that $D^2_{\theta,t}U_s$ is too. Applying once again the Malliavin derivative operator to (5.11) and using the second point in Lemma 21 as well as (5.20), we deduce for $\theta < t \leq s$,

$$\begin{aligned} D^2_{\theta,t}U_s &= \int_t^s (\beta \circ G^{-1})''(U_r)D_t U_r D_\theta U_r dr + \int_t^s (\beta \circ G^{-1})'(U_r) D^2_{\theta,t} U_r dr \\ &= \int_t^s e^{\int_r^s (\beta \circ G^{-1})'(U_v)dv} (\beta \circ G^{-1})''(U_r) D_t U_r D_\theta U_r dr \\ &= \int_t^s (\beta \circ G^{-1})''(U_r) D_r U_s D_t U_r D_\theta U_r dr. \end{aligned}$$

Further calculations yield the following expression

$$\begin{aligned} \left(\beta \circ G^{-1}\right)''(x) &= \left(\sigma\left(\frac{[\sigma,b]'}{\sigma} - \frac{[\sigma,b]\,\sigma'}{\sigma^2}\right) - \frac{1}{2}\left(\sigma''\sigma\right)'\sigma\right) \circ G^{-1}(x) \\ &= \left(\frac{[\sigma,[\sigma,b]]}{\sigma} - \frac{1}{2}\left(\sigma''\sigma\right)'\sigma\right) \circ G^{-1}(x). \end{aligned}$$

Using hypotheses (**H8**), (**H9**), the fact that $D_a U_b > 0$ for $a < b$ and (5.13), we immediatly obtain for $\theta < t \leq s$,

$$0 \leq \sigma(X_s) D^2_{\theta,t}U_s \leq \frac{2\left(M_{dl} + \frac{1}{2}M_{(\sigma''\sigma)'}M_\sigma\right)}{2M_l + M_\sigma M_{\sigma''}} \left[e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(2s-\theta-t)} - e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(s-\theta)}\right]. \tag{5.32}$$

Combining (5.30) and (5.32), it is clear that there exists a positive constant M_{D^2X} such that $0 \leq D^2_{\theta,t}X_s \leq M_{D^2X}$ with, for $\theta < t \leq s$,

$$\begin{aligned} M_{D^2X} = &\left(M_{\sigma'}M_\sigma + \frac{2\left(M_{dl} + \frac{1}{2}M_{(\sigma''\sigma)'}M_\sigma\right)}{2M_l + M_\sigma M_{\sigma''}}\right) e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(2s-t-\theta)} \\ &- \frac{2\left(M_{dl} + \frac{1}{2}M_{(\sigma''\sigma)'}M_\sigma\right)}{2M_l + M_\sigma M_{\sigma''}} e^{\left(M_l + \frac{M_\sigma M_{\sigma''}}{2}\right)(s-\theta)}. \end{aligned}$$

We will now address the second part of the Proposition, i.e., the inequalities on $D^2_{\theta,t}Y_s$. Let $\theta < t \leq s$. Applying once more the Malliavin derivative operator to $D_\theta Y_s$ in (5.18) and using the second point in Lemma 21, since f does not depend on Z we obtain, for $0 \leq \theta < t \leq s \leq T$,

$$\begin{aligned} D^2_{\theta,t}Y_s = &\phi'\left(X_T\right) D^2_{\theta,t}X_T + \phi''(X_T) D_\theta X_T D_t X_T - \int_s^T D^2_{\theta,t} Z_r dW_r \\ &+ \int_s^T \Big\{ f_{xx}\left(X_r, Y_r\right) D_\theta X_r D_t X_r + f_x\left(X_r, Y_r\right) D^2_{\theta,t} X_r \\ &\quad + f_{yx}\left(X_r, Y_r\right)\left(D_\theta Y_r D_t X_r + D_\theta X_r D_t Y_r\right) \\ &\quad + f_{yy}\left(X_r, Y_r\right) D_\theta Y_r D_t Y_r + f_y\left(X_r, Y_r\right) D^2_{\theta,t} Y_r \Big\} dr. \end{aligned}$$

Since $D^2_{\theta,t}Y_s$ solves a linear equation and is $\mathcal{F}_s$-measurable, we have that, for $0 \leq \theta < t \leq s \leq T$,

$$\begin{aligned} D^2_{\theta,t}Y_s =& \mathbf{E}\Big(e^{\int_s^T f_y(X_r,Y_r)dr} \{ \phi'(X_T) D^2_{\theta,t}X_T + \phi''(X_T) D_\theta X_T D_t X_T \} \Big| \mathcal{F}_s \Big) \\ &+ \mathbf{E}\Big(\int_s^T e^{\int_s^r f_y(X_u,Y_u)du} \Big\{ f_{xx}(X_r,Y_r) D_\theta X_r D_t X_r + f_x(X_r,Y_r) D^2_{\theta,t} X_r \\ &\quad + f_{yx}(X_r,Y_r)(D_\theta Y_r D_t X_r + D_\theta X_r D_t Y_r) + f_{yy}(X_r,Y_r) D_\theta Y_r D_t Y_r \Big\} dr \Big| \mathcal{F}_s \Big). \end{aligned}$$

Since $\sigma \geq c > 0$, (5.23) proves $D_u X_v \geq 0$ for $u \leq v$. Furthermore, (5.19) and (**H10**)–(**H11**) prove that $D_u Y_v \geq 0$ for $u \leq v$. Since (**H8**)–(**H11**) imply (**H5**)–(**H7**), the results in (5.17) and (5.23) remain valid. Thus, we immediatly obtain the positivity and an upper bound for $D^2_{\theta,t}Y_s$. This concludes the proof. ■

Proof of Theorem 16 : The outline of the proof is as follows : using a representation of Z, we compute its Malliavin derivative and show that under the hypotheses of Subsection 5.4.4, it is strictly bounded away from zero. This allows us to conclude using Proposition 3. We begin by giving a representation of Z. However, we do not use the one from [PP92] in terms of gradient, that is $Z_t = \sigma(X_t)(\nabla X_t)^{-1} \nabla Y_t$, but rather use the fact that Z_t can be represented by use of the Clark-Ocone formula. Indeed, by the uniqueness of the solution (Y, Z), Z_t can be written as

$$Z_t = \mathbf{E}\left(D_t \phi(X_T) + D_t \int_0^T f(X_s, Y_s)\, ds \Big| \mathcal{F}_t \right) \in \mathbb{D}^{1,2}. \tag{5.33}$$

Using this fact, we get for $t \in [0, T]$

$$Z_t = \mathbf{E}\left(\phi'(X_T) D_t X_T + \int_t^T \{ f_x(X_s,Y_s) D_t X_s + f_y(X_s,Y_s) D_t Y_s \} ds | \mathcal{F}_t \right).$$

Let $\theta \leq t$. We use both points of Lemma 21 and Proposition 6 in order to calculate the first order Malliavin derivative of Z_t. This leads, for $\theta \leq t$:

$$\begin{aligned} D_\theta Z_t =& \mathbf{E}\Big(\phi''(X_T) D_\theta X_T D_t X_T + \phi'(X_T) D^2_{\theta,t} X_T \\ &+ \int_t^T \Big\{ f_{xx}(X_s,Y_s) D_\theta X_s D_t X_s + f_{yx}(X_s,Y_s)(D_\theta Y_s D_t X_s + D_\theta X_s D_t Y_s) \\ &\quad + f_{yy}(X_s,Y_s) D_\theta Y_s D_t Y_s + f_x(X_s,Y_s) D^2_{\theta,t} X_s + f_y(X_s,Y_s) D^2_{\theta,t} Y_s \Big\} ds \Big| \mathcal{F}_t \Big). \end{aligned} \tag{5.34}$$

We now need to bound from above each summand of this expression ; in what follows, c and C denote strictly positive constants that may vary from line to line. Recall that under the assumptions (**H8**)–(**H11**) using (5.22) and (5.19), we deduce that $D_u X_v \geq c > 0$ and $D_u Y_v \geq 0$ for $u \leq v$. The hypothesis (**H10**) on ϕ (along with (**H8**) and (**H9**) on the diffusion X), (5.22) and Proposition 6 ensure that there exist strictly positive constants such that $0 < c \leq \phi''(X_T) D_\theta X_T D_t X_T \leq C$ and $0 \leq \phi'(X_T) D^2_{\theta,t} X_T \leq C$. Using hypothesis (**H11**) on f and its derivatives and Proposition 6 again allows us to bound the remaining terms in (5.34) by positive constants, i.e.

$$\begin{aligned} &0 \leq f_x(X_s,Y_s) D^2_{\theta,t} X_s \leq C, \quad 0 \leq f_y(X_s,Y_s) D^2_{\theta,t} Y_s \leq C, \\ &0 \leq f_{xx}(X_s,Y_s) D_\theta X_s D_t X_s \leq C, \quad 0 \leq f_{yy}(X_s,Y_s) D_\theta Y_s D_t Y_s \leq C, \\ &0 \leq f_{xy}(X_s,Y_s) D_\theta X_s D_t Y_s \leq C, \quad 0 \leq f_{yx}(X_s,Y_s) D_\theta Y_s D_t X_s \leq C. \end{aligned}$$

Gathering all of these immediatly gives us the existence of two strictly positive constants m_{DZ} and M_{DZ} such that for $0<\theta<t\leq T$, $\mathbb{P}-a.s$,

$$0<m_{DZ}\leq D_\theta Z_t\leq M_{DZ}. \tag{5.35}$$

Write $D_\bullet Z_t=\Phi^\bullet_{Z_t}(W)$ with a measurable function $\Phi^\bullet_{Z_t}:\mathbb{R}^{L^2([0,T])}\longrightarrow L^2([0,T])$. Then (5.35) yields, for $\theta<t$, $0<m_{DZ}\leq\Phi^\theta_{Z_t}(W)\leq M_{DZ}$. As previously done, define $\widetilde{\Phi^{\bullet,u}_{Z_t}}(W)=\Phi^\bullet_{Z_t}(e^{-u}W+\sqrt{1-e^{-2u}}W')$ for $u\in[0,+\infty[$. Using (5.35), it is clear that, for $\theta<t$, we have for $u\in[0,+\infty)$, $0<m_{DZ}\leq\widetilde{\Phi^{\theta,u}_{Z_t}}(W)\leq M_{DZ}$. Combining the bounds on $\Phi^\theta_{Z_t}$ and $\widetilde{\Phi^{\theta,u}_{Z_t}}$ yields, for $\theta<t$ and $u\in[0,+\infty)$,

$$0<m^2_{DZ}\leq\Phi^\theta_{Z_t}(W)\widetilde{\Phi^{\theta,u}_{Z_t}}(W)\leq M^2_{DZ}. \tag{5.36}$$

Finally, let

$$\begin{aligned} g(z)&=\int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'(\langle\Phi^\bullet_{Z_t}(W),\widetilde{\Phi^{\bullet,u}_{Z_t}}(W)\rangle_{L^2([0,T])})\Big|Z_t-\mathbf{E}(Z_t)=z\right)du\\ &=\int_0^\infty e^{-u}\mathbf{E}\left(\mathbf{E}'\left(\int_0^t\Phi^\theta_{Z_t}(W)\widetilde{\Phi^{\theta,u}_{Z_t}}(W)d\theta\right)\Big|Z_t-\mathbf{E}(Z_t)=z\right)du. \end{aligned}$$

The bounds obtained in (5.36) immediatly yield $0<m^2_{DZ}t\leq g(z)\leq M^2_{DZ}t$. Thus, Proposition 3 concludes the proof of Theorem 16. ■

Remark 20. *Theorem 16 has been proved under a set of hypotheses (those of Subsection 5.4.4) based on the fact that σ is positive. The case where σ is negative was included neither in the proof nor in the hypotheses for the sake of clarity and readability of the paper. However, as already mentioned in Remark 12, this case can be addressed (without any further difficulties) by using the following transformations : $\sigma\to\tilde\sigma:=-\sigma$ and $W\to\tilde W:=-W$. After performing those tranformations, it suffices to consider $(\tilde X,\tilde Y,\tilde Z)=(X,Y,-Z)$ to be the solution of*

$$\begin{cases} d\tilde X_t= & b\left(\tilde X_t\right)dt+\tilde\sigma\left(\tilde X_t\right)d\tilde W_t\\ d\tilde Y_t= & \phi\left(\tilde X_T\right)+\int_t^T f\left(\tilde X_r,\tilde Y_r\right)dr-\int_t^T\tilde Z_r d\tilde W_r \end{cases}$$

This brings the problem back to the set of hypotheses of Subsection 5.4.4 and it can be dealt with using the techniques presented in the last section.

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